J. Math. Anal. Appl. 368 (2010) 658–676

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Asymptotic analysis of the differences between the Stokes–Darcy system with different interface conditions and the Stokes–Brinkman system ☆

Nan Chen [a,*], Max Gunzburger [b], Xiaoming Wang [c]

[a] School of Mathematical Sciences, Fudan University, Shanghai, China, 200433
[b] Department of Scientific Computing, Florida State University, Tallahassee, FL 32306-4120, United States
[c] Department of Mathematics, Florida State University, Tallahassee, FL 32306-4120, United States

A R T I C L E   I N F O

Article history:
Received 31 August 2009
Available online 18 February 2010
Submitted by W. Layton

Keywords:
Stokes–Darcy equations
Stokes–Brinkman equations
Beavers–Joseph condition
Beavers–Joseph–Saffman–Jones condition

A B S T R A C T

We consider the coupling of the Stokes and Darcy systems with different choices for the interface conditions. We show that, comparing results with those for the Stokes–Brinkman equations, the solutions of Stokes–Darcy equations with the Beavers–Joseph interface condition in the one-dimensional and quasi-two-dimensional (periodic) cases are more accurate than are those obtained using the Beavers–Joseph–Saffman–Jones interface condition and that both of these are more accurate than solutions obtained using a zero tangential velocity interface condition. The zero tangential velocity interface condition is in turn more accurate than the free-slip interface boundary condition. We also prove that the summation of the quasi-two-dimensional solutions converge so that the conclusions are also valid for the two-dimensional case.

1. Introduction

Karst aquifers are among the most important type of groundwater systems. They are mostly made up of a porous medium, referred to as the matrix, that contains a network of fissures and conduits that are the major underground highways for water transport. The matrix holds water while in conduits, one has a free flow. Despite the fact that fissures and conduits occupy less space compared to the matrix, they play an essential role in the transport of fluid and contaminants in karst aquifers. Neglecting or not properly accounting for the flow in conduits and fissures and especially the exchange of fluid between them and the matrix can lead to inaccuracies.

Considerable effort has been directed at modeling and simulating the interaction between the confined flow in the matrices and the free flow in the conduit. The Navier–Stokes equations or their linearized counterpart, the Stokes equations, are widely used to describe the free flow in the conduit whereas Darcy's law is chosen to model the confined flow in the matrix. For connecting the components of the coupled Navier–Stokes–Darcy or Stokes–Darcy systems, two interface conditions are well accepted: the continuity of the normal velocity across the interface which is a consequence of the conservation of mass, and the balance of the stress force normal to the interface. Additional interface condition(s) is needed in order to close the system; the Beavers–Joseph interface condition [2] is regarded as perhaps providing the most faithful accounting of what happens at the matrix-conduit interface; there is abundant empirical evidence to support this claim. In the Beavers–Joseph interface condition, the tangential component of the stress force of the flow in the conduit at the interface is proportional to the jump in the tangential velocity across the interface. Unfortunately, from a mathematical point of view,

☆ Research supported in part by the U.S. National Science Foundation under grant number CMG DMS-0620035, the National Science Foundation of China under the grant number NSFC-10871050, and the Chinese Scholarship Council.

* Corresponding author.
E-mail addresses: chennan@fudan.edu.cn (N. Chen), gunzburg@fsu.edu (M. Gunzburger), wxm@math.fsu.edu (X. Wang).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.02.022

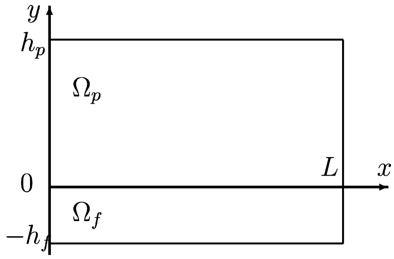

Fig. 1. The conduit (free flow) and matrix (porous media) domains Ω_f and Ω_p, respectively.

the Beavers–Joseph interface condition poses some difficulties because this condition makes an indefinite contribution to the total energy budget. Consequently, many simplified versions of this interface condition have emerged, among which the Beavers–Joseph–Saffman–Jones interface condition [13,14,19] is widely used; in this condition, the contribution of the tangential velocity in the porous media is neglected. As a result, the total energy budget is dissipative and hence analyses are substantially facilitated.[1] Despite the convenience for mathematical analysis, models using the Beavers–Joseph–Saffman–Jones interface condition can lead to an inaccurate accounting of the exchange of fluid between the matrix and conduit. A third choice for the remaining interface conditions is provided in [7] (see also [8,12]); there, the tangential velocity of the fluid in the conduit is set to be zero at the interface. A fourth candidate is discussed in [6] where the free-slip condition at the interface is proposed for the fluid flow in the conduit. These are even greater simplifications of the Beavers–Joseph interface condition and also further simplify mathematical and numerical analyses.

We need a reference solution to use to examine the differences resulting from the four choices of interface conditions within the Stokes–Darcy model. For this purpose, we replace the Darcy system with the Brinkman system as the model for the flow in the matrix. The Brinkman system is the extension of Darcy's law when boundary layer regions cannot be neglected. In fact, Neale and Nader showed in [18] the equivalency between Darcy–Stokes and Brinkman–Stokes velocities at the interface when the Brinkman viscosity is related to the Beavers–Joseph–Saffman parameter α for the simple case of one-dimensional shear flow. Note that it is well known that, as the Darcy number goes to zero, the differential equations of the Brinkman model reduce to those of the Darcy model [16]. Thus, the central question we address is the connection between the interface conditions of the Brinkman model with those corresponding to the four choices for the Stokes–Darcy model.

In this paper, we identify a non-dimensional parameter ϵ which is given by the square root of the ratio of the permeability to the porosity divided by a typical length scale in the porous media. We then perform an asymptotic analyses with respect to ϵ of the Stokes–Darcy model with four choices for the interface conditions. We use the Stokes–Brinkman model as the reference model to effect comparisons.

We should mention that the Beavers–Joseph–Saffman–Jones interface condition has been rigorously validated in the sense that, under appropriate assumptions, the solution of the Stokes–Darcy system with that interface condition is asymptotically the leading order of the solution of the Stokes equations in both the conduit and pore regions at small Darcy number; see [10,11]. Those results are complementary to our results because our work indicates that the Beavers–Joseph interface condition provides better approximations to the Brinkman–Stokes model at small Darcy number than does the Beavers–Joseph–Saffman–Jones model, but the correction is of lower order. Note, however, that the correction could be large in absolute value for not too small values of the Darcy number, a case that may be of interest in some applications such as metallic foams.

The paper is organized as follows. In Section 2, we provide the Stokes–Brinkman equations and also the Stokes–Darcy equations along with the four choices for the interface conditions. Section 3 is devoted to the one-dimensional case in which the tangential velocities only depend on horizontal variable and the normal velocities are identically zero. In Section 3, we also consider the fourth choice of interface condition, i.e., free-slip interface condition. In Section 4, we discuss the quasi-two-dimensional case in which the velocities depend on both the horizontal and vertical variables but are of special form. This is followed by a convergence theorem in Section 5 that gives us full two-dimensional solutions. In Section 6, we examine a separate issue by showing that the advective term is small for both the Brinkman or Darcy equations so that the linearized models can be regarded as valid approximations. Finally, in Section 7, we provide some concluding remarks.

2. The Stokes–Brinkman and Stokes–Darcy models

We start with a full description of the two models we consider. The two-dimensional conduit domain $\Omega_f = [0, L] \times [-h_f, 0]$ is occupied by a free flow; the two-dimensional matrix $\Omega_p = [0, L] \times [0, h_p]$ is occupied by a porous media, where L is order h_p. In this paper, we take $L = h_p$. We consider functions that are periodic in the horizontal variables with period h_p. Because the conduit occupies a much smaller space relative to the matrix, $h_p \gg h_f > 0$; see Fig. 1. We also denote

[1] Recently, in [3,4], the mathematical difficulties have been overcome and analyses and numerical analyses of the Stokes–Darcy model with the Beavers–Joseph interface condition have been provided.

by $\vec{u}_f$, p_f, $\vec{f}_f$ and $\vec{u}_p$, p_p, $\vec{f}_p$ the fluid velocity, kinematic pressure, and external body force in Ω_f and Ω_p, respectively; ν denotes the kinematic viscosity, n the porosity, Π the permeability, and $\mathbb{D}(\vec{u}) = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T)$ the deformation tensor. The relationship between porosity and permeability is given by $\Pi = \Pi_0 n^3/(1-n)^2$, where Π_0 is the typical permeability; see, e.g., [17]. We assume that n is a constant and that the flows in both the conduit and matrix are incompressible. The stress tensor is denoted by $\mathbb{T}(\vec{u}, p) = -p\mathbb{I} + 2\nu\mathbb{D}(\vec{u})$, where $\mathbb{I}$ denotes the identity tensor.

The original Brinkman equation is given by (see [5])

$$-2\widetilde{\nu}\nabla\cdot\mathbb{D}(\vec{u}_p) + \frac{\nu n}{\Pi}\vec{u}_p + n\nabla p_p = \vec{f}_p, \tag{1}$$

where $\widetilde{\nu}$ denotes the effective viscosity which can be different from ν. According to [18], the effective viscosity and the viscosity are related through

$$\sigma^2 = \frac{\widetilde{\nu}}{\nu}. \tag{2}$$

Dividing both sides of Eq. (1) by σ^2 yields

$$-2\nu\nabla\cdot\mathbb{D}(\vec{u}_p) + \frac{\nu n}{\Pi\sigma^2}\vec{u}_p + \frac{n}{\sigma^2}\nabla p_p = \frac{1}{\sigma^2}\vec{f}_p. \tag{3}$$

Here we assume that the effective viscosity and the viscosity are the same, i.e., $\sigma^2 = 1$, as is usually done in practice and analysis. The steady-state Stokes–Brinkman model for coupled conduit-matrix flows then takes the form

$$\begin{cases} -2\nu\nabla\cdot\mathbb{D}(\vec{u}_f) + \nabla p_f = \vec{f}_f, \qquad \nabla\cdot\vec{u}_f = 0, & \text{in } \Omega_f, \\ -2\nu\nabla\cdot\mathbb{D}(\vec{u}_p) + \frac{\nu n}{\Pi}\vec{u}_p + n\nabla p_p = \vec{f}_p, \qquad \nabla\cdot\vec{u}_p = 0, & \text{in } \Omega_p. \end{cases} \tag{4}$$

The original Brinkman model can be recovered easily by replacing n with $\frac{n}{\sigma^2}$, and replacing $\vec{f}_p$ by $\frac{\vec{f}_p}{\sigma^2}$.

At the interface between the conduit and matrix domains, two sets of interface conditions are widely used. One is the standard continuity of the velocity and the stress force across the interface, i.e.,

$$\vec{u}_f = \vec{u}_p, \qquad \left(-p_f\mathbb{I} + 2\nu\mathbb{D}(\vec{u}_f)\right)\cdot\vec{n} = \left(-np_p\mathbb{I} + 2\nu\mathbb{D}(\vec{u}_p)\right)\cdot\vec{n}. \tag{5}$$

The other is continuity of the velocity, all velocity derivatives,[2] and the pressure across the interface proposed by Le Bars and Worster [15], i.e.,

$$\vec{u}_f = \vec{u}_p, \qquad \nabla\vec{u}_f = \nabla\vec{u}_p, \qquad p_f = p_p. \tag{6}$$

These two types of interface boundary conditions reduce to the same ones in the one-dimensional case whereas we use the latter one when dealing with two-dimensional systems.

We introduce the non-dimensional variables utilizing typical reference quantities in the matrix:

$$x' = \frac{x}{h_p}, \qquad p' = \frac{p}{\frac{\nu h_p U}{\Pi_0}}, \qquad u' = \frac{u}{U}, \qquad f' = \frac{f}{\frac{\nu U}{\Pi_0}}. \tag{7}$$

With these notations in hand, the non-dimensional form of the Brinkman equation is

$$-\frac{\nu U}{h_p^2}\Delta'\vec{u}'_p + \frac{\nu n U}{\Pi}\vec{u}'_p + \frac{n\nu h_p U}{h_p\Pi_0}\nabla' p_p = \frac{\nu U}{\Pi_0}\vec{f}'_p,$$

which is, after dropping the primes,

$$-\frac{\Pi_0}{h_p^2}\Delta\vec{u}_p + \frac{n\Pi_0}{\Pi}\vec{u}_p + n\nabla p_p = \vec{f}_p. \tag{8}$$

We introduce a non-dimensional parameter, the Darcy number $\mathrm{Da} = \Pi_0/h_p^2$. When the Darcy number goes to zero, the Brinkman equation (8) reduces to the Darcy equation

$$\vec{u}_p = -\frac{\Pi}{\Pi_0}\left(\nabla p_p - \frac{1}{n}\vec{f}_p\right).$$

[2] When considering the effective viscosity, this should be replaced by $\nu\nabla\vec{u}_f = \widetilde{\nu}\nabla\vec{u}_p$.

Similarly, the non-dimensional form of Stokes equation is

$$-\frac{\Pi_0}{h_p^2}\Delta\vec{u}_f + \nabla p_p = \vec{f}_f. \tag{9}$$

Although (9) also contain the Darcy number, the order of the term $\Delta\vec{u}_f$ is not $O(1)$ since the non-dimensionalization is based on reference quantities in the porous media. Therefore, this term cannot be dropped.

Collecting the above results and returning to the dimensional form of the equations, we have the Stokes–Darcy system

$$\begin{cases} -2\nu\nabla\cdot\mathbb{D}(\vec{u}_f) + \nabla p_f = \vec{f}_f, \qquad \nabla\cdot\vec{u_f} = 0, \quad \text{in } \Omega_f, \\ \vec{u}_p = -\dfrac{\Pi}{\nu}\left(\nabla p_p - \dfrac{1}{n}\vec{f}_p\right), \qquad \nabla\cdot\vec{u_p} = 0, \qquad \text{in } \Omega_p \end{cases} \tag{10}$$

for the coupled conduit-matrix flows. The Stokes–Darcy equations are supplemented by periodic boundary condition in the horizontal direction and no-penetration and free-slip boundary condition at the top and bottom for simplicity[3]

$$\frac{\partial u_{f1}}{\partial y} = u_{f2} = 0, \quad \text{on } y = -h_f, \tag{11}$$

$$\frac{\partial u_{p1}}{\partial y} = u_{p2} = 0, \quad \text{on } y = h_p. \tag{12}$$

The system is also augmented by the interface conditions

$$\vec{u}_f\cdot\vec{n}_{pf} = \vec{u}_p\cdot\vec{n}_{pf}, \tag{13a}$$

$$-\vec{n}_{pf}\cdot\big(\mathbb{T}(\vec{u}_f, p_f)\vec{n}_{pf}\big) = g(h_p - y), \tag{13b}$$

$$-\vec{\tau}_{pf}\cdot\big(\mathbb{T}(\vec{u}_f, p_f)\vec{n}_{pf}\big) = \alpha\frac{\nu}{\sqrt{\Pi}}\vec{\tau}_{pf}\cdot(\vec{u}_f - \vec{u}_p), \tag{13c}$$

where $h_p = y + (p_p/(\rho g))$ denotes the hydraulic head, α a constant, $\vec{n}_{pf}$ a unit vector normal to the interface, and $\vec{\tau}_{pf}$ a unit vector tangent to the interface.[4]

The interface conditions (13) for the Stokes–Darcy model are known as the *Beavers–Joseph conditions* [2]. The first two interface conditions in (13) are quite natural; (13a) guarantees the conservation of mass and (13b) the continuity of the normal stress[5] across the interface Γ. On the other hand, (13c) is a not a statement of continuity of the tangential stress or the velocity derivative components across the interface Γ.[6] Near the interface Γ, a boundary layer may form in the matrix; this boundary layer is not resolved by the Darcy equations. Thus, (13c) models the jump in the tangential stress across that boundary layer. In particular, it says that the tangential stress of the conduit flow at the interface Γ is proportional to the jump in the tangential velocities across the boundary layer, in the limit that the boundary layer thickness vanishes; see [2] for details. The value of the parameter α depends on the properties of the porous material as well as the geometrical setting of the coupled problem; it also can be used as a model tuning parameter.

A widely accepted simplification of the Beavers–Joseph conditions is the *Beavers–Joseph–Saffman–Jones conditions* [13,19] in which the term $\vec{\tau}_{pf}\cdot\vec{u}_p$ on the right-hand side of (13c) is neglected so that equation is replaced by

$$-\tau_{pf}\cdot\big(\mathbb{T}(\vec{u}_f, p_f)\vec{n}_{pf}\big) = \alpha\frac{\nu}{\sqrt{\Pi}}\tau_{pf}\cdot\vec{u}_f. \tag{10c$'$}$$

A further simplification [7] of the Beavers–Joseph conditions is to ignore the left-hand side, i.e., the tangential stress force, in (10c′) so that, as a result, the tangential velocity of the fluid in the conduit is set to zero, i.e., we have

$$\tau_{pf}\cdot\vec{u}_f = 0. \tag{10c$''$}$$

In the sequel, for simplicity, we refer to (10c″) as the *zero tangential velocity* interface condition, even though it only sets the tangential velocity in the conduit to zero.

Yet another simplification [6] of the Beavers–Joseph conditions is to ignore the right-hand side, i.e., setting $\alpha = 0$ in (10c′) so that, as a result, velocity of the fluid in the conduit satisfies the free-slip condition at the interface, i.e., we have

$$\tau_{pf}\cdot\big(\mathbb{T}(\vec{u}_f, p_f)\vec{n}_{pf}\big) = 0. \tag{10c$'''$}$$

For simplicity, we refer to (10c‴) as the *free-slip* interface condition in the sequel.

[3] The free-slip condition at the bottom may be replaced by the no-slip condition. This is necessary for well posedness if we adopt the simplified free-slip (10c‴) at the interface between the conduit and matrix.

[4] In the set up of Fig. 1, we can choose $\vec{n}_{pf}$ and $\vec{\tau}_{pf}$ to be the unit vectors in the y and x directions, respectively.

[5] The stress force (or force due to stress) acting on a surface in the flow having the unit normal vector $\vec{n}$ is given by $\mathbb{T}(\vec{u}, p)\vec{n}$ so that the normal and tangential stresses on that surface are given by $\vec{n}\cdot(\mathbb{T}(\vec{u}, p)\vec{n})$ and $\vec{\tau}\cdot(\mathbb{T}(\vec{u}, p)\vec{n})$, respectively.

[6] Thus, (13c) is not obtainable through a direct reduction of (5) or (6).

In the karst aquifer setting, the non-dimensional parameter $\epsilon = \sqrt{\Pi/n}/h_p$ is usually small. We investigate the asymptotic behavior of the velocities with respect to ϵ and then compare the solutions of the Stokes–Brinkman model (4)–(6) with those of the three Stokes–Darcy models, i.e., the model (10)–(13) with the Beavers–Joseph condition, the model (10), (13a), (13b), and (10c′) with the Beavers–Joseph–Saffman–Jones condition, the model (10), (13a), (13b), and (10c″) with the zero tangential velocity interface condition, and the model (10), (13a), (13b), and (10c‴) with the free-slip interface condition.[7]

3. One-dimensional flows

In the one-dimensional case, we assume that the normal velocities are identically zero and that the tangential velocities only depend on y so that we have the ansatz for the velocities, pressures, and body forces given by

$$\begin{cases} \vec{u}_f = \big(u_f(y), 0\big), \\ \vec{u}_p = \big(u_p(y), 0\big), \end{cases} \qquad \begin{cases} p_f \equiv 0, \\ p_p \equiv 0, \end{cases} \qquad \begin{cases} \vec{f}_f = \big(f_f(y), 0\big), \\ \vec{f}_p = \big(f_p(y), 0\big). \end{cases} \tag{14}$$

3.1. Asymptotic solutions of the Stokes–Brinkman system

We first focus on the asymptotic behavior of solutions of the Stokes–Brinkman system (4)–(6).

Lemma 3.1. *The exact solution of Stokes–Brinkman system* (4)–(6) *in the one-dimensional case is given by*

$$u_f = -\frac{1}{\nu}\int_0^y\int_0^t f_f(s)\,ds\,dt - \frac{y}{\nu}\int_{-h_f}^0 f_f(s)\,ds + \sqrt{\frac{\Pi}{n}}\frac{2\widetilde{A} - C_2\cosh(\sqrt{\frac{n}{\Pi}}h_p)}{\sinh(\sqrt{\frac{n}{\Pi}}h_p)},$$

$$u_p = \sqrt{\frac{\Pi}{n}}\frac{\widetilde{A}\cosh(\sqrt{\frac{n}{\Pi}}y) - C_2\cosh(\sqrt{\frac{n}{\Pi}}(y-h_p))}{\sinh(\sqrt{\frac{n}{\Pi}}h_p)} + \frac{1}{2\nu}\sqrt{\frac{\Pi}{n}}\int_0^y f_p(s)\big(e^{-\sqrt{\frac{n}{\Pi}}(y-s)} - e^{\sqrt{\frac{n}{\Pi}}(y-s)}\big)\,ds, \tag{15}$$

where

$$\widetilde{A} = \frac{1}{2\nu}\int_0^{h_p} f_p(s)\big(e^{-\sqrt{\frac{n}{\Pi}}(h_p-s)} + e^{\sqrt{\frac{n}{\Pi}}(h_p-s)}\big)\,ds, \qquad C_2 = -\frac{1}{\nu}\int_{-h_f}^0 f_f(s)\,ds.$$

Proof. With the ansatz (14) in hand, the coupled Stokes–Brinkman equations (4) reduce to

$$-\nu u_f'' = f_f, \quad y \in (-h_f, 0) \quad \text{and} \quad -\nu u_p'' + \frac{\nu n}{\Pi}u_p = f_p, \quad y \in (0, h_p),$$

the general solutions of which are given by

$$\begin{cases} u_f = -\dfrac{1}{\nu}\displaystyle\int_0^y\int_0^t f_f(s)\,ds\,dt + C_1 + C_2 y, \\ u_p = \dfrac{1}{2\nu}\sqrt{\dfrac{\Pi}{n}}\displaystyle\int_0^y f_p(s)\big(e^{-\sqrt{\frac{n}{\Pi}}(y-s)} - e^{\sqrt{\frac{n}{\Pi}}(y-s)}\big)\,ds + C_3 e^{\sqrt{\frac{n}{\Pi}}y} + C_4 e^{-\sqrt{\frac{n}{\Pi}}y}, \end{cases} \tag{16}$$

respectively. The interface conditions at $y = 0$ reduce to $u_f(0) = u_p(0)$ and $u_f'(0) = u_p'(0)$ so that

$$C_1 = C_3 + C_4 \quad \text{and} \quad C_2 = \sqrt{\frac{n}{\Pi}}(C_3 - C_4). \tag{17}$$

The free-slip boundary conditions at $y = h_p$ and $y = -h_f$ reduce to $u_f'(-h_f) = 0$ and $u_p'(h_p) = 0$, respectively, so that, together with (16), we have

[7] In this last case, the free-slip boundary condition at the bottom boundary $y = -h_f$ should be replaced by the no-slip condition in order to ensure well posedness.

$$\begin{cases} \dfrac{1}{\nu}\displaystyle\int_{-h_f}^{0} f_f(s)\,ds + C_2 = 0, \\ -\dfrac{1}{2\nu}\displaystyle\int_{0}^{h_p} f_p(s)\big(e^{-\sqrt{\frac{n}{\Pi}}(h_p-s)} + e^{\sqrt{\frac{n}{\Pi}}(h_p-s)}\big)\,ds + \sqrt{\dfrac{n}{\Pi}}\big(C_3 e^{\sqrt{\frac{n}{\Pi}}h_p} - C_4 e^{-\sqrt{\frac{n}{\Pi}}h_p}\big) = 0. \end{cases}$$

Together with (17), we then have

$$C_1 = \sqrt{\frac{\Pi}{n}}\frac{2\widetilde{A} - C_2(e^{\sqrt{\frac{n}{\Pi}}h_p} + e^{-\sqrt{\frac{n}{\Pi}}h_p})}{e^{\sqrt{\frac{n}{\Pi}}h_p} - e^{-\sqrt{\frac{n}{\Pi}}h_p}}, \qquad C_2 = -\frac{1}{\nu}\int_{-h_f}^{0} f_f(s)\,ds,$$

$$C_3 = \sqrt{\frac{\Pi}{n}}\frac{\widetilde{A} - C_2 e^{-\sqrt{\frac{n}{\Pi}}h_p}}{e^{\sqrt{\frac{n}{\Pi}}h_p} - e^{-\sqrt{\frac{n}{\Pi}}h_p}}, \qquad C_4 = \sqrt{\frac{\Pi}{n}}\frac{\widetilde{A} - C_2 e^{\sqrt{\frac{n}{\Pi}}h_p}}{e^{\sqrt{\frac{n}{\Pi}}h_p} - e^{-\sqrt{\frac{n}{\Pi}}h_p}}.$$

Inserting these into the general solutions (16) completes the proof. □

Lemma 3.2. *Let $\epsilon = \sqrt{\Pi/n}/h_p$. The asymptotic solution of the Stokes–Brinkman system* (4)–(6) *in the one-dimensional case is given by*

$$u_f \sim \frac{1}{\nu}\left(-\int_0^y\int_0^t f_f(s)\,ds\,dt - y\int_{-h_f}^{0} f_f(s)\,ds + \epsilon h_p\int_{-h_f}^{0} f_f(s)\,ds + \epsilon^2 h_p^2 f_p(0) + \epsilon^3 h_p^3 f_p'(0) + \epsilon^4 h_p^4 f_p''(0)\right),$$

$$u_p \sim \frac{1}{\nu}\left(\epsilon^3 h_p^3 f_p'(0) + \epsilon h_p\int_{-h_f}^{0} f_f(s)\,ds\right)e^{-\frac{1}{\epsilon}\frac{y}{h_p}} + \frac{1}{\nu}\big(\epsilon^2 h_p^2 f_p(y) + \epsilon^4 h_p^4 f_p''(y)\big) - \frac{1}{\nu}\epsilon^3 h_p^3 f'(h_p)e^{-\frac{1}{\epsilon}(1-\frac{y}{h_p})}. \tag{18}$$

Proof. Note that the first two terms in the equation for u_f in (15) do not depend on ϵ. In light of the fact that $\frac{1}{\epsilon} \gg 1$, by dividing both the numerator and denominator of the third term by $e^{\frac{1}{\epsilon}}$ and dropping the exponentially small term $e^{-\frac{2}{\epsilon}}$, we obtain

$$\sqrt{\frac{\Pi}{n}}\frac{2\widetilde{A} - C_2(e^{\sqrt{\frac{n}{\Pi}}h_p} + e^{-\sqrt{\frac{n}{\Pi}}h_p})}{e^{\sqrt{\frac{n}{\Pi}}h_p} - e^{-\sqrt{\frac{n}{\Pi}}h_p}} \sim \epsilon h_p\big(2\widetilde{A}e^{-\frac{1}{\epsilon}} - C_2\big).$$

Expanding $\widetilde{A}e^{-\frac{1}{\epsilon}}$ with respect to ϵ yields

$$\begin{aligned} \widetilde{A}e^{-\frac{1}{\epsilon}} &= \frac{1}{2\nu}\int_0^{h_p} f_p(s)\big(e^{-\frac{1}{\epsilon}(2-\frac{s}{h_p})} + e^{-\frac{1}{\epsilon}\frac{s}{h_p}}\big)\,ds \\ &= \frac{1}{2\nu}\left(\int_0^{h_p} f_p(s)e^{-\frac{1}{\epsilon}\frac{s}{h_p}}\,ds + e^{-\frac{2}{\epsilon}}\int_0^{h_p} f_p(s)e^{\frac{1}{\epsilon}\frac{s}{h_p}}\,ds\right) \\ &\sim \frac{1}{2\nu}\big(\epsilon h_p f_p(0) + \epsilon^2 h_p^2 f_p'(0) + \epsilon^3 h_p^3 f_p''(0) - 2\epsilon^2 h_p^2 f_p'(h_p)e^{-\frac{1}{\epsilon}}\big). \end{aligned} \tag{19}$$

Thus, we obtain the first relation in (18).

In the same spirit, the first term of u_p in (15) can be reduced to

$$\sqrt{\frac{\Pi}{n}}\frac{\widetilde{A}\cosh(\sqrt{\frac{n}{\Pi}}y) - C_2\cosh(\sqrt{\frac{n}{\Pi}}(y-h_p))}{\sinh(\sqrt{\frac{n}{\Pi}}h_p)} \sim \epsilon h_p\widetilde{A}e^{-\frac{1}{\epsilon}}\big(e^{-\frac{1}{\epsilon}\frac{y}{h_p}} + e^{\frac{1}{\epsilon}\frac{y}{h_p}}\big) - C_2 e^{-\frac{1}{\epsilon}\frac{y}{h_p}}$$

by dropping the exponentially small factors. With (19) in hand, it is easy to show that

$$\epsilon h_p \widetilde{A} e^{-\frac{1}{\epsilon}}\left(e^{-\frac{1}{\epsilon}\frac{y}{h_p}} + e^{\frac{1}{\epsilon}\frac{y}{h_p}}\right) - \frac{\epsilon h_p}{2\nu} e^{\frac{1}{\epsilon}\frac{y}{h_p}} \int_0^y f_p(s) e^{-\frac{1}{\epsilon}\frac{s}{h_p}}$$

$$= \frac{\epsilon h_p}{2\nu}\left(\int_0^{h_p} f_p(s) e^{-\frac{1}{\epsilon}\frac{s}{h_p}}\,ds + e^{-\frac{2}{\epsilon}}\int_0^{h_p} f_p(s) e^{\frac{1}{\epsilon}\frac{s}{h_p}}\,ds\right)\left(e^{-\frac{1}{\epsilon}\frac{y}{h_p}} + e^{\frac{1}{\epsilon}\frac{y}{h_p}}\right) - \frac{\epsilon h_p}{2\nu} e^{\frac{1}{\epsilon}\frac{y}{h_p}} \int_0^y f_p(s) e^{-\frac{1}{\epsilon}\frac{s}{h_p}}$$

$$\sim \frac{\epsilon h_p}{2\nu} e^{\frac{1}{\epsilon}\frac{y}{h_p}} \int_y^{h_p} f_p(s) e^{-\frac{1}{\epsilon}\frac{s}{h_p}}\,ds + \frac{\epsilon h_p}{2\nu} e^{-\frac{1}{\epsilon}\frac{y}{h_p}} \int_0^{h_p} f_p(s) e^{-\frac{1}{\epsilon}\frac{s}{h_p}}\,ds + \frac{\epsilon h_p}{2\nu} e^{-\frac{2}{\epsilon}} e^{\frac{1}{\epsilon}\frac{y}{h_p}} \int_0^{h_p} f_p(s) e^{\frac{1}{\epsilon}\frac{s}{h_p}}\,ds$$

$$\sim -\frac{\epsilon^2 h_p^2}{2\nu} e^{-\frac{1}{\epsilon}(1-\frac{y}{h_p})}\left(f_p(h_p) + \epsilon h_p f_p'(h_p)\right) + \frac{\epsilon^2 h_p^2}{2\nu}\left(f_p(y) + \epsilon h_p f_p'(y) + \epsilon^2 h_p^2 f_p''(y)\right)$$
$$+ \frac{\epsilon^2 h_p^2}{2\nu} e^{-\frac{1}{\epsilon}\frac{y}{h_p}}\left(f_p(0) + \epsilon h_p f_p'(0)\right) + \frac{\epsilon^2 h_p^2}{2\nu} e^{-\frac{1}{\epsilon}(1-\frac{y}{h_p})}\left(f_p(h_p) - \epsilon h_p f_p'(h_p)\right)$$
$$\sim -\frac{\epsilon^3 h_p^3}{\nu} e^{-\frac{1}{\epsilon}(1-\frac{y}{h_p})} f_p'(h_p) + \frac{\epsilon^2 h_p^2}{2\nu}\left(f_p(y) + \epsilon h_p f_p'(y) + \epsilon^2 h_p^2 f_p''(y)\right) + \frac{\epsilon^2 h_p^2}{2\nu} e^{-\frac{1}{\epsilon}\frac{y}{h_p}}\left(f_p(0) + \epsilon h_p f_p'(0)\right). \tag{20}$$

For the second term of u_p in (15), we repeat the same procedure as for (19) to obtain

$$\frac{\epsilon h_p}{2\nu} e^{-\frac{1}{\epsilon}\frac{y}{h_p}} \int_0^y f_p(s) e^{\frac{1}{\epsilon}\frac{s}{h_p}}\,ds \sim \frac{1}{2\nu}\Big(-\left(\epsilon^2 h_p^2 f_p(0) - \epsilon^3 h_p^3 f_p'(0)\right) e^{-\frac{1}{\epsilon}\frac{y}{h_p}}$$
$$+ \left(\epsilon^2 h_p^2 f_p(y) - \epsilon^3 h_p^3 f_p'(y) + \epsilon^4 h_p^4 f_p''(y)\right)\Big). \tag{21}$$

Combining with (20) yields

$$u_p \sim \frac{1}{\nu}\left(\epsilon^3 h_p^3 f_p'(0) + \epsilon h_p \int_{-h_f}^0 f_f(s)\,ds\right) e^{-\frac{1}{\epsilon}\frac{y}{h_p}} + \frac{1}{\nu}\left(\epsilon^2 h_p^2 f_p(y) + \epsilon^4 h_p^4 f_p''(y)\right) - \frac{1}{\nu}\epsilon^3 h_p^3 f'(h_p) e^{-\frac{1}{\epsilon}(1-\frac{y}{h_p})}$$

which leads to the second relation in (18). □

3.2. Asymptotic solutions of the Stokes–Darcy system

Next, we focus on deriving the asymptotic solution of one-dimensional Stokes–Darcy system (10), (13a), and (13b) with the Beavers–Joseph interface condition (13c).

Lemma 3.3. *Let $\epsilon = \sqrt{\Pi/n}/h_p$. The asymptotic solution of the one-dimensional Stokes–Darcy equations with the Beavers–Joseph interface condition* (13c) *is given by*

$$\begin{cases} u_{p,BJ}^0 = \dfrac{\epsilon^2}{\nu} h_p^2 f_p(y), \\ u_{f,BJ}^0 = -\dfrac{1}{\nu}\displaystyle\int_0^y\int_0^t f_f(s)\,ds\,dt - \dfrac{y}{\nu}\int_{-h_f}^0 f_f(s)\,ds + \dfrac{\sqrt{n}}{\alpha}\dfrac{\epsilon}{\nu} h_p \int_{-h_f}^0 f_f(s)\,ds + \dfrac{\epsilon^2}{\nu} h_p^2 f_p(0). \end{cases} \tag{22}$$

Proof. In the one-dimensional case, the Stokes–Darcy equations (10) reduce to

$$-\nu u_f'' = f_f, \quad y \in (-h_f, 0) \quad \text{and} \quad u_p = \frac{\Pi}{\nu n} f_p, \quad y \in (0, h_p), \tag{23}$$

the general solutions of which are given by

$$u_{f,BJ}^0 = -\frac{1}{\nu}\int_0^y\int_0^t f_f(s)\,ds\,dt + C_1 + C_2 y \quad \text{and} \quad u_{p,BJ}^0 = \frac{\Pi}{\nu n} f_p(y). \tag{24}$$

Clearly, the first two interface boundary conditions (13a) and (13b) are satisfied automatically, while the Beavers–Joseph condition (13c) reduces to

$$-\nu \frac{\partial u^0_{f,BJ}}{\partial y}\bigg|_{y=0} = \frac{\alpha\nu}{\sqrt{\Pi}}\left(u^0_{f,BJ} - u^0_{p,BJ}\right)\bigg|_{y=0}.$$

To determine the coefficients, we impose this condition and the free-slip boundary condition at $y=-h_f$ on the general solution (23) to obtain

$$C_1 = \frac{\sqrt{\Pi}}{\alpha}\frac{A}{\nu} + \frac{\Pi}{\nu n} f_p(0) \quad \text{and} \quad C_2 = -\frac{A}{\nu}, \quad A = \int_{-h_f}^{0} f_f(s)\,ds.$$

Accordingly, we arrive at the asymptotic solution (22) of Stokes–Darcy equations with Beavers–Joseph interface conditions by setting $\epsilon = \sqrt{\Pi/n}/h_p$. $\square$

Using the same arguments, one obtains the asymptotic solution of the Stokes–Darcy system (10), (13a), and (13b) with the Beavers–Joseph–Saffman–Jones interface condition (10c′) and the zero tangential velocity interface condition (10c″).

Lemma 3.4. *Let $\epsilon = \sqrt{\Pi/n}/h_p$. The asymptotic solution of the one-dimensional Stokes–Darcy system with the Beavers–Joseph–Saffman–Jones interface condition* (10c′) *is given by*

$$\begin{cases} u^0_{p,BJSJ} = \dfrac{\epsilon^2}{\nu} h_p^2 f_p(y), \\ u^0_{f,BJSJ} = -\dfrac{1}{\nu}\displaystyle\int_0^y\int_0^t f_f(s)\,ds\,dt - \dfrac{y}{\nu}\int_{-h_f}^{0} f_f(s)\,ds + \dfrac{\sqrt{n}}{\alpha}\dfrac{\epsilon}{\nu} h_p \int_{-h_f}^{0} f_f(s)\,ds. \end{cases} \tag{25}$$

Lemma 3.5. *Let $\epsilon = \sqrt{\Pi/n}/h_p$. The asymptotic solution of the one-dimensional Stokes–Darcy system with zero tangential velocity interface condition* (10c″) *is given by*

$$\begin{cases} u^0_{p,Q} = \dfrac{\epsilon^2}{\nu} h_p^2 f_p(y), \\ u^0_{f,Q} = -\dfrac{1}{\nu}\displaystyle\int_0^y\int_0^t f_f(s)\,ds\,dt - \dfrac{y}{\nu}\int_{-h_f}^{0} f_f(s)\,ds. \end{cases} \tag{26}$$

Remark. In the one-dimensional case, one observes the following from (18), (22), (25), and (26).

1. The optimal choice[8] of α is $\alpha = \sqrt{n}$.
2. Solutions with both Beavers–Joseph and Beavers–Joseph–Saffman–Jones interface conditions have low sensitivity on α for α near this optimal value or larger. This can be seen via a direct differentiation together with the Kozeny–Carman formula relating permeability and porosity [1]. This low sensitivity is observed in numerical simulations [9].
3. The leading-order term for the velocities for the Stokes–Brinkman and Stokes–Darcy systems are both of $O(1/\nu)$ in the conduit and are both of $O(\epsilon^2/\nu)$ in the matrix.
4. The velocity for the Stokes–Brinkman system contains boundary layers in the matrix near both $y=0$ and $y=h_p$, the order being $O(\epsilon/\nu)$ and $O(\epsilon^3/\nu)$, respectively.

We have also deduced the asymptotic behavior of solutions of the Stokes–Darcy system with the free-slip interface condition (10c‴); however, for well posedness, in this case we have to replace the free-slip boundary condition at the bottom boundary $y=-h_f$ by the physical no-slip boundary condition. In the case of the no-slip boundary condition at $y=-h_f$, we have also deduced the asymptotic behavior of solutions of the Stokes–Darcy system with the three other interface conditions, i.e., with (13c) or (10c′) or (10c″), and also for the Stokes–Brinkman system; in all cases, the leading order behavior does not change from that for the free-slip boundary condition at $y=-h_f$, so we do not report on these results here. The calculations for the free-slip interface condition (10c‴) are very much the same as those for the other interface conditions, so that we also do not report on them here. We merely include the implications resulting from the use of the interface condition (10c‴) in the comparisons made in Section 3.3.

[8] When considering the effective viscosity in the Brinkman system, the corresponding optimal choice of α is $\alpha = \sigma\sqrt{n} = \sqrt{n\tilde{\nu}/\nu}$.

3.3. Comparison of solutions of the Stokes–Brinkman and Stokes–Darcy systems

With (18), (22), (25), and (26) in hand, we can compare solutions of the Stokes–Darcy system with different interface conditions with solutions of the Stokes–Brinkman system under the optimal choice of $\alpha = \sqrt{n}$.

Proposition 3.6. *For the one-dimensional case, the difference between the velocity in the conduit for the Stokes–Brinkman system and the Stokes–Darcy system with the Beavers–Joseph interface condition is of $O(\epsilon^3/\nu)$. That difference is of $O(\epsilon^2/\nu)$ for the Beavers–Joseph–Saffman–Jones interface condition and the difference is of $O(\epsilon/\nu)$ for the zero tangential velocity interface condition.*

Proposition 3.7. *For the one-dimensional case, the difference between the velocity in the matrix for the Stokes–Brinkman system and the Stokes–Darcy equations with either the Beavers–Joseph, the Beavers–Joseph–Saffman–Jones, or the zero tangential velocity interface conditions are all of $O(\epsilon^4/\nu)$.*

From Proposition 3.6, we see that, when comparing with solutions of the Stokes–Brinkman system, the solution of the Stokes–Darcy system with the Beavers–Joseph interface condition fits better than does the solution obtained using the Beavers–Joseph–Saffman–Jones interface condition and both are better fits that are solutions obtained using the tangential velocity interface condition. Note that the special case of f_f and f_p being constants is essentially the same as that studied in [19].

The free-slip interface condition (10c‴) formally corresponds to the case $\alpha = 0$ in the Beavers–Joseph interface condition (13c). Comparisons of asymptotic solutions with those for the Stokes–Brinkman system are given in the following proposition.

Proposition 3.8. *For the one-dimensional case, the difference between the velocity in the conduit for the Stokes–Brinkman system and the Stokes–Darcy system with the free-slip interface condition is of $O(1/\nu)$. The difference between the velocities in the matrix is of $O(\epsilon^4/\nu)$ as it is for the other interface conditions.*

From Propositions 3.6 and 3.8, we see that, when comparing with solutions of the Stokes–Brinkman system, the solution of the Stokes–Darcy system with the Beavers–Joseph interface condition fits better than does the solution obtained using the Beavers–Joseph–Saffman–Jones interface condition and both are better fits than are solutions obtained using the tangential velocity interface condition and all three are better fits than are solution obtained using the zero-slip interface condition.[9] Note that the special case of f_f and f_p being constants is essentially the same as that studied in [19].

4. Quasi-two-dimensional flows

We now consider solutions and body forces that depend on both x and y, assuming periodicity in the horizontal direction, i.e., we invoke the ansatz

$$\vec{u}_f = \sum_{k=-K}^{K} \vec{u}_{f,k} = \sum_{k=-K}^{K} \big(u_{f,1,k}(y), u_{f,2,k}(y)\big) e^{\frac{i2\pi kx}{h_p}},$$

$$\vec{f}_f = \sum_{k=-K}^{K} \vec{f}_{f,k} = \sum_{k=-K}^{K} \big(f_{f,1,k}(y), f_{f,2,k}(y)\big) e^{\frac{i2\pi kx}{h_p}},$$

$$\vec{u}_p = \sum_{k=-K}^{K} \vec{u}_{p,k} = \sum_{k=-K}^{K} \big(u_{p,1,k}(y), u_{p,2,k}(y)\big) e^{\frac{i2\pi kx}{h_p}},$$

$$\vec{f}_p = \sum_{k=-K}^{K} \vec{f}_{p,k} = \sum_{k=-K}^{K} \big(f_{p,1,k}(y), f_{p,2,k}(y)\big) e^{\frac{i2\pi kx}{h_p}},$$

where the integer k denotes the wave number. Here, we also make the assumption that the Fourier decomposition only contains a finite number of modes. Because solutions and data are real functions, we only need to consider $k \geqslant 0$. To simplify notation, in the sequel we set $\widetilde{k} = \frac{2\pi k}{h_p}$.

[9] Due to its poor performance as an approximation to the Beavers–Joseph interface condition and to save space, we do not consider the zero-slip interface condition any further.

4.1. Solutions of the Stokes–Brinkman system

We start by deriving the general solution of the Stokes–Brinkman system (4) for each fixed k.

Lemma 4.1. *For the Stokes–Brinkman system* (4) *and for each fixed k, the normal velocity in the conduit takes the form*

$$u_{f,2,k}e^{i\widetilde{k}x} = \left(C_1e^{\widetilde{k}y} + C_2e^{-\widetilde{k}y} + C_3ye^{\widetilde{k}y} + C_4ye^{-\widetilde{k}y} + u_{fs,k}\right)e^{i\widetilde{k}x}, \tag{27}$$

where the coefficients C_i, $i = 1, 2, 3, 4$, are to be determined and the particular solution $u_{fs,k}$ is given by

$$\begin{aligned} u_{fs,k} = -\frac{1}{4\nu}\Bigg(& e^{\widetilde{k}y}\int_0^y e^{-\widetilde{k}s}\frac{1+\widetilde{k}s}{\widetilde{k}^3}F(s)\,ds + e^{-\widetilde{k}y}\int_0^y e^{\widetilde{k}s}\frac{-1+\widetilde{k}s}{\widetilde{k}^3}F(s)\,ds \\ & - ye^{\widetilde{k}y}\int_0^y e^{-\widetilde{k}s}\frac{1}{\widetilde{k}^2}F(s)\,ds - ye^{-\widetilde{k}y}\int_0^y e^{\widetilde{k}s}\frac{1}{\widetilde{k}^2}F(s)\,ds\Bigg), \end{aligned} \tag{28}$$

where

$$F(s) = \widetilde{k}^2\left(f_{f,2,k}(s) + \frac{i}{\widetilde{k}}f'_{f,1,k}(s)\right). \tag{29}$$

Proof. Because $\vec{u}_{f,k}$ is solenoidal, there exists a stream function ψ such that $\vec{u}_{f,k} = (-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x})$. Let $\psi = \frac{1}{i\widetilde{k}}u_{f,2,k}(y)e^{i\widetilde{k}x}$. Then,

$$\vec{u}_{f,k} = \left(-u'_{f,2,k}(y), i\widetilde{k}u_{f,2,k}(y)\right)\frac{1}{i\widetilde{k}}e^{i\widetilde{k}x}. \tag{30}$$

The pressure in the Stokes equation for the conduit can be eliminated by taking the curl of that equation, resulting in

$$-\nu\Delta_2\psi = \nabla\times\vec{f}_{f,k} = \left(i\widetilde{k}f_{f,2,k}(y) - f'_{f,1,k}(y)\right)e^{i\widetilde{k}x}$$

which, together with (30), leads to the ordinary differential equation

$$\frac{-\nu}{i\widetilde{k}}\left(u^{(4)}_{f,2,k} - 2\widetilde{k}^2u''_{f,2,k} + \widetilde{k}^4u_{f,2,k}\right)e^{i\widetilde{k}x} = \nabla\times\vec{f}_{f,k}. \tag{31}$$

Let

$$F := -i\widetilde{k}\nabla\times\vec{f}_{f,k}e^{-i\widetilde{k}x} = \widetilde{k}^2f_{f,2,k}(y) + i\widetilde{k}f'_{f,1,k}(y) = \widetilde{k}^2\left(f_{f,2,k}(y) + \frac{i}{\widetilde{k}}f'_{f,1,k}(y)\right).$$

Then, (31) becomes

$$u^{(4)}_{f,2,k} - 2\widetilde{k}^2u''_{f,2,k} + \widetilde{k}^4u_{f,2,k} = \frac{1}{\nu}F \tag{32}$$

for which we have the solution (27)–(28). □

Using the same argument, we select $\phi = \frac{1}{i\widetilde{k}}u_{p,2,k}(y)e^{i\widetilde{k}x}$ such that

$$\vec{u}_p = \left(-\frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial x}\right) = \left(-u'_{p,2,k}(y), i\widetilde{k}u_{p,2,k}(y)\right)\frac{1}{i\widetilde{k}}e^{i\widetilde{k}x} \tag{33}$$

and define

$$G(y) := \widetilde{k}^2\left(f_{p,2,k}(y) + \frac{i}{\widetilde{k}}f'_{p,1,k}(y)\right). \tag{34}$$

Then, we have the following result.

Lemma 4.2. *For the Stokes–Brinkman system* (4) *and for each fixed $\widetilde{k}$, the normal velocity in the matrix takes the form*

$$u_{p,2,k}e^{i\widetilde{k}x} = \left(C_5e^{\widetilde{k}y} + C_6e^{-\widetilde{k}y} + C_7e^{\sqrt{\widetilde{k}^2+\frac{n}{\Pi}}\,y} + C_8e^{-\sqrt{\widetilde{k}^2+\frac{n}{\Pi}}\,y} + u_{ps,k}\right)e^{i\widetilde{k}x}, \tag{35}$$

where the coefficients C_i, $i = 5, 6, 7, 8$, are to be determined and the particular solution $u_{ps,k}$ is given by

$$u_{ps,k} = -\frac{1}{2\nu}\left(e^{\widetilde{k}y}\int_0^y \frac{1}{\widetilde{k}\frac{n}{\Pi}}e^{-\widetilde{k}s}G(s)\,ds - e^{-\widetilde{k}y}\int_0^y \frac{1}{\widetilde{k}\frac{n}{\Pi}}e^{\widetilde{k}s}G(s)\,ds - e^{\sqrt{\widetilde{k}^2+\frac{n}{\Pi}}\,y}\int_0^y \frac{1}{\sqrt{\widetilde{k}^2+\frac{n}{\Pi}}\,\frac{n}{\Pi}}e^{-\sqrt{\widetilde{k}^2+\frac{n}{\Pi}}\,s}G(s)\,ds\right.$$
$$\left. + e^{-\sqrt{\widetilde{k}^2+\frac{n}{\Pi}}\,y}\int_0^y \frac{1}{\sqrt{\widetilde{k}^2+\frac{n}{\Pi}}\,\frac{n}{\Pi}}e^{\sqrt{\widetilde{k}^2+\frac{n}{\Pi}}\,s}G(s)\,ds\right).$$

It remains to determine the coefficients C_i, $i = 1, 2, \ldots, 8$. We set, as in the one-dimensional case, $\epsilon = \sqrt{\Pi/n}/h_p$ and let $E = \sqrt{\widetilde{k}^2 + (nh_p^2/\Pi)}/h_p$.

Lemma 4.3. *For the Stokes–Brinkman system* (4)–(6) *and for each fixed $\widetilde{k}$, the coefficients C_i, $i = 1, 2, \ldots, 8$, are the solution of the linear system*

$$\begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 & -1 & -1 \\ \widetilde{k} & -\widetilde{k} & 1 & 1 & -\widetilde{k} & \widetilde{k} & -E & E \\ \widetilde{k}^2 & \widetilde{k}^2 & 2\widetilde{k} & -2\widetilde{k} & -\widetilde{k}^2 & -\widetilde{k}^2 & -E^2 & -E^2 \\ e^{-\widetilde{k}h_f} & e^{\widetilde{k}h_f} & -h_f e^{-\widetilde{k}h_f} & -h_f e^{\widetilde{k}h_f} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\widetilde{k}h_p} & e^{-\widetilde{k}h_p} & e^{Eh_p} & e^{-Eh_p} \\ 0 & 0 & -2 & -2 & -\frac{1}{\widetilde{k}\Pi} & \frac{1}{\widetilde{k}\Pi} & 0 & 0 \\ \widetilde{k}^2 e^{-\widetilde{k}h_f} & \widetilde{k}^2 e^{\widetilde{k}h_f} & (-\widetilde{k}^2 h_f + 2\widetilde{k})e^{-\widetilde{k}h_f} & (-\widetilde{k}^2 h_f - 2\widetilde{k})e^{\widetilde{k}h_f} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widetilde{k}^2 e^{\widetilde{k}h_p} & \widetilde{k}^2 e^{-\widetilde{k}h_p} & E^2 e^{Eh_p} & E^2 e^{-Eh_p} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{pmatrix},$$

where

$$f_1 = f_2 = f_3 = 0, \qquad f_4 = -u_{fs,k}(-h_f), \qquad f_5 = -u_{ps,k}(-h_p),$$
$$f_6 = -\frac{i}{\nu\widetilde{k}}\left(f_{f,1,k}(0) - \frac{1}{n}f_{p,1,k}(0)\right), \qquad f_7 = -u''_{fs,k}(-h_f), \qquad f_8 = -u''_{ps,k}(-h_p).$$

Proof. Using (30) and (33), the first two interface conditions in (6) for the Stokes–Brinkman system reduce to

$$u_{f,2,k}(0) = u_{p,2,k}(0), \qquad u'_{f,2,k}(0) = u'_{p,2,k}(0), \qquad u''_{f,2,k}(0) = u''_{p,2,k}(0)$$

which imply that

$$C_1 + C_2 = C_5 + C_6 + C_7 + C_8, \tag{C1}$$
$$\widetilde{k}C_1 - \widetilde{k}C_2 + C_3 + C_4 = \widetilde{k}C_5 - \widetilde{k}C_6 + EC_7 - EC_8, \tag{C2}$$
$$\widetilde{k}^2C_1 + \widetilde{k}^2C_2 + 2\widetilde{k}C_3 - 2\widetilde{k}C_4 = \widetilde{k}^2C_5 + \widetilde{k}^2C_6 + E^2C_7 + E^2C_8. \tag{C3}$$

Next, the Stokes–Brinkman equations (4) imply that the pressure on the two sides of the interface is given by

$$\begin{aligned} \left.\frac{\partial p_{f,k}}{\partial x}\right|_{y=0} &= f_{f,1,k}(0)e^{i\widetilde{k}x} + 2\nu\left.\left(\frac{\partial^2 u_{f,1,k}}{\partial x^2} + \frac{1}{2}\left(\frac{\partial^2 u_{f,1,k}}{\partial y^2} + \frac{\partial^2 u_{f,2,k}}{\partial x \partial y}\right)\right)\right|_{y=0} \\ &= f_{f,1,k}(0)e^{i\widetilde{k}x} + \nu\left(-2\widetilde{k}^2C_3 - 2\widetilde{k}^2C_4\right)\frac{1}{i\widetilde{k}}e^{i\widetilde{k}x}, \\ \left.\frac{\partial p_{p,k}}{\partial x}\right|_{y=0} &= \frac{1}{n}f_{p,1,k}(0)e^{i\widetilde{k}x} + \frac{2\nu}{n}\left(\frac{\partial^2 u_{p,1,k}}{\partial x^2} + \frac{1}{2}\left(\frac{\partial^2 u_{p,1,k}}{\partial y^2} + \frac{\partial^2 u_{p,2,k}}{\partial x \partial y}\right)\right) - \left.\frac{\nu}{\Pi}u_{p,1,k}\right|_{y=0} \\ &= \frac{1}{n}f_{p,1,k}(0)e^{i\widetilde{k}x} + \frac{\nu}{n}\left(\widetilde{k}^2(EC_7 - EC_8) - \left(E^3C_7 - E^3C_8\right)\right)\frac{1}{i\widetilde{k}}e^{i\widetilde{k}x} + \frac{\nu}{\Pi}(\widetilde{k}C_5 - \widetilde{k}C_6 + EC_7 - EC_8)\frac{1}{i\widetilde{k}}e^{i\widetilde{k}x}. \end{aligned}$$

Let $p_{f,k}|_{x=0,\,y=0} = p_{f,k}(0,0)$ and $p_{p,k}|_{x=0,\,y=0} = p_{p,k}(0,0)$. Integrating p_f and p_p from 0 to x yields

$$p_{f,k} = f_{f,1,k}(0)\frac{1}{i\widetilde{k}}\left(e^{i\widetilde{k}x} - 1\right) + 2\nu(C_3 + C_4)\left(e^{i\widetilde{k}x} - 1\right) + p_{f,k}(0,0) \tag{36}$$

and

$$\begin{aligned}
p_{p,k} &= f_{p,1,k}(0)\frac{1}{ni\widetilde{k}}\left(e^{i\widetilde{k}x}-1\right) - \frac{\nu}{n}\frac{E(\widetilde{k}^2-E^2)}{\widetilde{k}^2}(C_7-C_8)\left(e^{i\widetilde{k}x}-1\right) \\
&\quad - \frac{\nu}{\Pi\widetilde{k}^2}(\widetilde{k}C_5-\widetilde{k}C_6+EC_7-EC_8)\left(e^{i\widetilde{k}x}-1\right)+p_{p,k}(0,0) \\
&= f_{p,1,k}(0)\frac{1}{ni\widetilde{k}}\left(e^{i\widetilde{k}x}-1\right) + \frac{\nu}{\Pi}\left(-\frac{1}{\widetilde{k}}C_5+\frac{1}{\widetilde{k}}C_6\right)\left(e^{i\widetilde{k}x}-1\right)+p_{p,k}(0,0). \qquad (37)
\end{aligned}$$

With (36) and (37) in hand, we set $x=0$ in the third interface condition in (6) (which holds for all x) to obtain $p_{p,k}(0,0)-p_{f,k}(0,0)=0$. Then,

$$-2C_3-2C_4-\frac{1}{\widetilde{k}\Pi}C_5+\frac{1}{\widetilde{k}\Pi}C_6=-\frac{i}{\nu\widetilde{k}}\left(f_{f,1,k}(0)-\frac{1}{n}f_{p,1,k}(0)\right). \qquad (C4)$$

Besides the interface boundary conditions, we impose free-slip boundary conditions at $-h_f$ and h_p:

$$u'_{f,1,k}(-h_f)=0, \qquad u'_{p,1,k}(h_p)=0, \qquad u_{f,2,k}(-h_f)=0, \qquad u_{p,2,k}(h_p)=0$$

so that

$$\widetilde{k}^2e^{-\widetilde{k}h_f}C_1+\widetilde{k}^2e^{\widetilde{k}h_f}C_2+\left(2\widetilde{k}-\widetilde{k}^2h_f\right)e^{-\widetilde{k}h_f}C_3-\left(2\widetilde{k}+\widetilde{k}^2h_f\right)e^{\widetilde{k}h_f}C_4=-u''_{fs,k}(-h_f), \qquad (C5)$$

$$\widetilde{k}^2e^{\widetilde{k}h_p}C_5+\widetilde{k}^2e^{-\widetilde{k}h_p}C_6+E^2e^{Eh_p}+E^2e^{-Eh_p}=-u''_{ps,k}(h_p), \qquad (C6)$$

$$e^{-\widetilde{k}h_f}C_1+e^{\widetilde{k}h_f}C_2-h_fe^{-\widetilde{k}h_f}C_3+h_fe^{\widetilde{k}h_f}C_3=-u_{fs,k}(-h_f), \qquad (C7)$$

$$e^{\widetilde{k}h_p}C_5+e^{-\widetilde{k}h_p}C_6+e^{Eh_p}C_7+e^{-Eh_p}C_8=-u_{ps,k}(h_p). \qquad (C8)$$

Consequently, combining (C1)–(C8) completes the proof. □

4.2. Solutions of the Stokes–Darcy systems

We next obtain the solutions of the Stokes–Darcy system (10) in the quasi-two-dimensional case. In the same spirit as for the Stokes–Brinkman system, we invoke the ansatz

$$\begin{aligned}
\vec{u}_f^0 &= \sum_{k=-K}^{K}\vec{u}_{f,k}^0=\sum_{k=-K}^{K}\left(u^0_{f,1,k}(y),u^0_{f,2,k}(y)\right)e^{i\widetilde{k}x}, \\
\vec{u}_p^0 &= \sum_{k=-K}^{K}\vec{u}_{p,k}^0=\sum_{k=-K}^{K}\left(u^0_{p,1,k}(y),u^0_{p,2,k}(y)\right)e^{i\widetilde{k}x}
\end{aligned}$$

for solutions of the Stokes–Darcy system. By selecting the streamfunctions $\psi=\frac{1}{i\widetilde{k}}u^0_{f,2,k}(y)e^{i\widetilde{k}x}$ and $\phi=\frac{1}{i\widetilde{k}}u^0_{p,2,k}(y)e^{i\widetilde{k}x}$, the velocities can be written as

$$\vec{u}^0_{f,k}=\left(-u^{0\prime}_{f,2,k}(y),i\widetilde{k}u^0_{f,2,k}(y)\right)\frac{1}{i\widetilde{k}}e^{i\widetilde{k}x}, \qquad \vec{u}^0_{p,k}=\left(-u^{0\prime}_{p,2,k}(y),i\widetilde{k}u^0_{p,2,k}(y)\right)\frac{1}{i\widetilde{k}}e^{i\widetilde{k}x}.$$

Using the same argument as for the Stokes–Brinkman system, we obtain the following result.

Lemma 4.4. *For the Stokes–Darcy system* (10) *and for each fixed $\widetilde{k}$, the normal velocity in the conduit and matrix are given by*

$$u^0_{f,2,k}e^{i\widetilde{k}x}=\left(C_1^0e^{\widetilde{k}y}+C_2^0e^{-\widetilde{k}y}+C_3^0ye^{\widetilde{k}y}+C_4^0ye^{-\widetilde{k}y}+u^0_{fs,k}\right)e^{i\widetilde{k}x} \qquad (38)$$

and

$$u^0_{p,2,k}e^{i\widetilde{k}x}=\left(C_5^0e^{\widetilde{k}y}+C_6^0e^{-\widetilde{k}y}+u^0_{ps,k}\right)e^{i\widetilde{k}x}, \qquad (39)$$

respectively, where the coefficients C_i^0, $i=1,2,\ldots,6$, are to be determined and the particular solutions $u^0_{fs,k}$ and $u^0_{ps,k}$ are given by

$$\begin{aligned}
u^0_{fs,k} &= -\frac{1}{4\nu}\left(e^{\widetilde{k}y}\int_0^y e^{-\widetilde{k}s}\frac{1+\widetilde{k}s}{\widetilde{k}^3}F(s)\,ds+e^{-\widetilde{k}y}\int_0^y e^{\widetilde{k}s}\frac{-1+\widetilde{k}s}{\widetilde{k}^3}F(s)\,ds\right. \\
&\quad \left. - ye^{\widetilde{k}y}\int_0^y e^{-\widetilde{k}s}\frac{1}{\widetilde{k}^2}F(s)\,ds-ye^{-\widetilde{k}y}\int_0^y e^{\widetilde{k}s}\frac{1}{\widetilde{k}^2}F(s)\,ds\right)
\end{aligned}$$

and

$$u^0_{ps,k} = -\frac{1}{2\nu}\frac{\Pi}{n}e^{\widetilde{k}y}\int_0^y \frac{1}{\widetilde{k}}e^{-\widetilde{k}s}G(s)\,ds + \frac{1}{2\nu}\frac{\Pi}{n}e^{-\widetilde{k}y}\int_0^y \frac{1}{\widetilde{k}}e^{\widetilde{k}s}G(s)\,ds,$$

respectively, where F and G are defined in (29) *and* (34)*, respectively.*

Lemma 4.5. *For the Stokes–Darcy system* (10), (13a), *and* (13b) *and for each fixed k, the coefficients C^0_i, $i = 1, 2, \dots, 6$, are the solution of the linear system*

$$\begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ -2\widetilde{k}^2 & 2\widetilde{k}^2 & 0 & 0 & 1/\Pi & -1/\Pi \\ \widetilde{k}^2 e^{-\widetilde{k}h_f} & \widetilde{k}^2 e^{\widetilde{k}h_f} & (2\widetilde{k} - h_f\widetilde{k}^2)e^{-\widetilde{k}h_f} & (-2\widetilde{k} - h_f\widetilde{k}^2)e^{\widetilde{k}h_f} & 0 & 0 \\ e^{-\widetilde{k}h_f} & e^{\widetilde{k}h_f} & -h_f e^{-\widetilde{k}h_f} & -h_f e^{\widetilde{k}h_f} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\widetilde{k}h_p} & e^{-\widetilde{k}h_p} \\ 2\widetilde{k}^2 + \frac{\widetilde{k}}{\epsilon h_p} & 2\widetilde{k}^2 - \frac{\widetilde{k}}{\epsilon h_p} & 2\widetilde{k} + \frac{1}{\epsilon h_p} & -2\widetilde{k}^2 + \frac{1}{\epsilon h_p} & -\frac{\widetilde{k}}{\epsilon h_p} & \frac{\widetilde{k}}{\epsilon h_p} \end{pmatrix} \begin{pmatrix} C^0_1 \\ C^0_2 \\ C^0_3 \\ C^0_4 \\ C^0_5 \\ C^0_6 \end{pmatrix} = \begin{pmatrix} g^0_1 \\ g^0_2 \\ g^0_3 \\ g^0_4 \\ g^0_5 \\ g^0_6 \end{pmatrix}$$

for the Beavers–Joseph interface condition (13c)*,*

$$\begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ -2\widetilde{k}^2 & 2\widetilde{k}^2 & 0 & 0 & 1/\Pi & -1/\Pi \\ \widetilde{k}^2 e^{-\widetilde{k}h_f} & \widetilde{k}^2 e^{\widetilde{k}h_f} & (2\widetilde{k} - h_f\widetilde{k}^2)e^{-\widetilde{k}h_f} & (-2\widetilde{k} - h_f\widetilde{k}^2)e^{\widetilde{k}h_f} & 0 & 0 \\ e^{-\widetilde{k}h_f} & e^{\widetilde{k}h_f} & -h_f e^{-\widetilde{k}h_f} & -h_f e^{\widetilde{k}h_f} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\widetilde{k}h_p} & e^{-\widetilde{k}h_p} \\ 2\widetilde{k}^2 + \frac{\widetilde{k}}{\epsilon h_p} & 2\widetilde{k}^2 - \frac{\widetilde{k}}{\epsilon h_p} & 2\widetilde{k} + \frac{1}{\epsilon h_p} & -2\widetilde{k}^2 + \frac{1}{\epsilon h_p} & 0 & 0 \end{pmatrix} \begin{pmatrix} C^0_1 \\ C^0_2 \\ C^0_3 \\ C^0_4 \\ C^0_5 \\ C^0_6 \end{pmatrix} = \begin{pmatrix} g^0_1 \\ g^0_2 \\ g^0_3 \\ g^0_4 \\ g^0_5 \\ g^0_6 \end{pmatrix}$$

for the Beavers–Joseph–Saffman–Jones interface condition, and (10c$'$)

$$\begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ -2\widetilde{k}^2 & 2\widetilde{k}^2 & 0 & 0 & 1/\Pi & -1/\Pi \\ \widetilde{k}^2 e^{-\widetilde{k}h_f} & \widetilde{k}^2 e^{\widetilde{k}h_f} & (2\widetilde{k} - h_f\widetilde{k}^2)e^{-\widetilde{k}h_f} & (-2\widetilde{k} - h_f\widetilde{k}^2)e^{\widetilde{k}h_f} & 0 & 0 \\ e^{-\widetilde{k}h_f} & e^{\widetilde{k}h_f} & -h_f e^{-\widetilde{k}h_f} & -h_f e^{\widetilde{k}h_f} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\widetilde{k}h_p} & e^{-\widetilde{k}h_p} \\ \widetilde{k} & -\widetilde{k} & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} C^0_1 \\ C^0_2 \\ C^0_3 \\ C^0_4 \\ C^0_5 \\ C^0_6 \end{pmatrix} = \begin{pmatrix} g^0_1 \\ g^0_2 \\ g^0_3 \\ g^0_4 \\ g^0_5 \\ g^0_6 \end{pmatrix}$$

for the zero tangential velocity interface condition (10c$''$)*, where*

$$g_1 = g_6 = 0, \qquad g_2 = \frac{i}{\nu}\left(f_{f,1,k}(0) - \frac{1}{n}f_{p,1,k}(0)\right),$$
$$g_3 = -u^{0\prime\prime}_{fs,k}(-h_f), \qquad g_4 = -u^0_{fs,k}(-h_f), \qquad g_5 = -u^0_{ps,k}(h_p).$$

Proof. Condition (13a) results in $u^0_{f,2,k}(0) = u^0_{p,2,k}(0)$ so that

$$C^0_1 + C^0_2 = C^0_5 + C^0_6. \tag{D1}$$

Normalized with $\rho = 1$ and written in component form, (13b) takes the form

$$-(0,1)\left[-\begin{pmatrix} p_{f,k} & 0 \\ 0 & p_{f,k} \end{pmatrix} + 2\nu \begin{pmatrix} \frac{\partial u^0_{f,1,k}}{\partial x} & \frac{1}{2}\left(\frac{\partial u^0_{f,1,k}}{\partial y} + \frac{\partial u^0_{f,2,k}}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u^0_{f,1,k}}{\partial y} + \frac{\partial u^0_{f,2,k}}{\partial x}\right) & \frac{\partial u^0_{f,2,k}}{\partial y} \end{pmatrix}\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix} = p_{p,k}$$

which implies

$$p_{f,k} - 2\nu\frac{\partial u^0_{f,2,k}}{\partial y} = p_{p,k}. \tag{40}$$

The Stokes equation in (10) implies

$$\frac{\partial p_{f,k}}{\partial x} = f_{f,1,k}(0) + 2\nu\left(\frac{\partial^2 u^0_{f,1,k}}{\partial x^2} + \frac{1}{2}\left(\frac{\partial^2 u^0_{f,1,k}}{\partial y^2} + \frac{\partial^2 u^0_{f,2,k}}{\partial x \partial y}\right)\right).$$

Let $p_{f,k}|_{x=0,\, y=0} = p_{f,k}(0,0)$ and $p_{p,k}|_{x=0,\, y=0} = p_{p,k}(0,0)$. Integrating from 0 to x yields

$$p_{f,k} = f_{f,1,k}(0)\frac{1}{i\widetilde{k}}\left(e^{i\widetilde{k}x} - 1\right) + 2\nu\left(C_3^0 + C_4^0\right)\left(e^{i\widetilde{k}x} - 1\right) + p_{f,k}(0,0). \tag{41}$$

On the other hand, the Darcy equation in (10) implies

$$\frac{\partial p_{p,k}}{\partial x} = \frac{1}{n} f_{p,1,k}(0) - \frac{\nu}{\Pi} u^0_{p,1,k} = \frac{1}{n} f_{p,1,k} e^{i\widetilde{k}x} + \frac{\nu}{\Pi}\left(\widetilde{k}C_5^0 - \widetilde{k}C_6^0\right)\frac{1}{i\widetilde{k}} e^{i\widetilde{k}x}.$$

Integrating from 0 to x results in

$$p_{p,k} = \frac{1}{i\widetilde{k}n} f_{p,1,k}(0)\left(e^{i\widetilde{k}x} - 1\right) - \frac{\nu}{\Pi}\frac{1}{\widetilde{k}}\left(C_5^0 - C_6^0\right)\left(e^{i\widetilde{k}x} - 1\right) + p_{p,k}(0,0). \tag{42}$$

Furthermore, it is obvious that

$$2\nu\frac{\partial u^0_{f,2,k}}{\partial y} = 2\nu\left(\widetilde{k}C_1^0 - \widetilde{k}C_2^0 + C_3^0 + C_4^0\right)e^{i\widetilde{k}x}. \tag{43}$$

Inserting (41)–(43) into (40) and setting $x = 0$ yields $p_{p,k}(0,0) - p_{f,k}(0,0) = 2\nu(\widetilde{k}C_1^0 - \widetilde{k}C_2^0 + C_3^0 + C_4^0)$. Elementary calculations show that (40) can be reduced to

$$-2\widetilde{k}^2 C_1^0 + 2\widetilde{k}^2 C_2^0 + \frac{1}{\Pi}C_5^0 - \frac{1}{\Pi}C_6^0 = \frac{i}{\nu}\left(f_{f,1,k}(0) - \frac{1}{n} f_{p,1,k}(0)\right). \tag{D2}$$

Written in component form, the Beavers–Joseph interface condition (13c) takes the form

$$-(1,0)\left[-\begin{pmatrix} p_{f,k} & 0 \\ 0 & p_{f,k}\end{pmatrix} + 2\nu\begin{pmatrix} \frac{\partial u^0_{f,1,k}}{\partial x} & \frac{1}{2}\left(\frac{\partial u^0_{f,1,k}}{\partial y} + \frac{\partial u^0_{f,2,k}}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u^0_{f,1,k}}{\partial y} + \frac{\partial u^0_{f,2,k}}{\partial x}\right) & \frac{\partial u^0_{f,2,k}}{\partial y}\end{pmatrix}\right]\begin{pmatrix}0\\1\end{pmatrix}$$
$$= \frac{\alpha\nu}{\sqrt{\Pi}}(1,0)\begin{pmatrix} u^0_{f,1,k} - u^0_{p,1,k} \\ u^0_{f,2,k} - u^0_{p,2,k}\end{pmatrix},$$

i.e.,

$$-\nu\left(\frac{\partial u^0_{f,1,k}}{\partial y} + \frac{\partial u^0_{f,2,k}}{\partial x}\right) = \frac{\alpha\nu}{\sqrt{\Pi}}\left(u^0_{f,1,k} - u^0_{p,1,k}\right).$$

We then have, after setting $\alpha = \sqrt{n}$,

$$\left(2\widetilde{k}^2 + \sqrt{\frac{n}{\Pi}}\widetilde{k}\right)C_1^0 + \left(2\widetilde{k}^2 - \sqrt{\frac{n}{\Pi}}\widetilde{k}\right)C_2^0 + \left(2\widetilde{k} + \sqrt{\frac{n}{\Pi}}\right)C_3^0 + \left(-2\widetilde{k} + \sqrt{\frac{n}{\Pi}}\right)C_4^0 - \sqrt{\frac{n}{\Pi}}\widetilde{k}C_5^0 + \sqrt{\frac{n}{\Pi}}\widetilde{k}C_6^0 = 0. \tag{D3}$$

Using the same argument for the Beavers–Joseph–Saffman–Jones interface boundary condition (10c′) leads to

$$\left(2\widetilde{k}^2 + \sqrt{\frac{n}{\Pi}}\widetilde{k}\right)C_1^0 + \left(2\widetilde{k}^2 - \sqrt{\frac{n}{\Pi}}\widetilde{k}\right)C_2^0 + \left(2\widetilde{k} + \sqrt{\frac{n}{\Pi}}\right)C_3^0 + \left(-2\widetilde{k} + \sqrt{\frac{n}{\Pi}}\right)C_4^0 = 0. \tag{D3′}$$

Similarly, for the zero tangential velocity interface condition (10c″), we have

$$\widetilde{k}C_1^0 - \widetilde{k}C_2^0 + C_3^0 + C_4^0 = 0. \tag{D3″}$$

We also impose the free-slip boundary conditions $u^{0\prime}_{f,1,k}(-h_f) = 0$ and $u^0_{f,2,k}(-h_f) = 0$ at $-h_f$ so that

$$\widetilde{k}^2 e^{-\widetilde{k}h_f}C_1^0 + \widetilde{k}^2 e^{\widetilde{k}h_f}C_2^0 + \left(2\widetilde{k} - \widetilde{k}^2 h_f\right)e^{-\widetilde{k}h_f}C_3^0 - \left(2\widetilde{k} + \widetilde{k}^2 h_f\right)e^{\widetilde{k}h_f}C_4^0 = -u^{0\prime\prime}_{fs,k}(-h_f), \tag{D4}$$

$$e^{-\widetilde{k}h_f}C_1^0 + e^{\widetilde{k}h_f}C_2^0 - h_f e^{-\widetilde{k}h_f}C_3^0 - h_f e^{\widetilde{k}h_f}C_4^0 = u^0_{fs,k}(-h_f) \tag{D5}$$

along with the no-flow condition $u^0_{p,2,k}(h_p) = 0$ across the boundary at h_p so that

$$e^{\widetilde{k}h_p}C_5^0 + e^{-\widetilde{k}h_p}C_6^0 = -u^0_{ps,k}(h_p). \tag{D6}$$

Combining (D1)–(D6), (D3′) and (D3″) completes the proof. □

4.3. Comparison of asymptotic solutions of the Stokes–Brinkman and Stokes–Darcy systems

With the coefficients solved using MATLAB, we have the following results.

Proposition 4.6. *The asymptotic solution of the normal velocity for the quasi-two-dimensional Stokes–Brinkman system is given by*

$$\begin{aligned} u_{f,2,k} &\sim C_{k,1}e^{i\widetilde{k}x}\frac{1}{\nu} + C_{k,2}e^{i\widetilde{k}x}\frac{\epsilon}{\nu} + C_{k,3}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + O\left(\epsilon^3/\nu\right), \\ u_{p,2,k} &\sim P_{k,1}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + P_{k,2}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + P_{k,4}e^{i\widetilde{k}x}\frac{\epsilon^4}{\nu} + O\left(\epsilon^5/\nu\right) + P_{k,5}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu}e^{-Ey} + P_{k,6}e^{i\widetilde{k}x}\frac{\epsilon^4}{\nu}e^{E(y-h_p)}, \end{aligned} \tag{44}$$

where the coefficients are listed in Appendix A.

Proposition 4.7. *The asymptotic solution of the normal velocity for the quasi-two-dimensional Stokes–Darcy system with the Beavers–Joseph interface condition is given by*

$$\begin{aligned} u^0_{f,2,k,BJ} &\sim C_{k,1}e^{i\widetilde{k}x}\frac{1}{\nu} + C_{k,2}e^{i\widetilde{k}x}\frac{\epsilon}{\nu} + C_{k,3}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + O\left(\epsilon^3/\nu\right), \\ u^0_{p,2,k,BJ} &\sim P_{k,1}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + P_{k,2}e^{i\widetilde{k}x}\frac{\epsilon^3}{\nu} + P_{k,3}e^{i\widetilde{k}x}\frac{\epsilon^4}{\nu} + O\left(\epsilon^6/\nu\right), \end{aligned} \tag{45}$$

with the Beavers–Joseph–Saffman–Jones interface condition by

$$\begin{aligned} u^0_{f,2,k,BJSJ} &\sim C_{k,1}e^{i\widetilde{k}x}\frac{1}{\nu} + C_{k,2}e^{i\widetilde{k}x}\frac{\epsilon}{\nu} + C_{k,4}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + O\left(\epsilon^3/\nu\right), \\ u^0_{p,2,k,BJSJ} &\sim P_{k,1}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + P_{k,2}e^{i\widetilde{k}x}\frac{\epsilon^3}{\nu} + P_{k,3}e^{i\widetilde{k}x}\frac{\epsilon^4}{\nu} + P_{k,7}e^{i\widetilde{k}x}\frac{\Pi\epsilon^2}{\nu} + O\left(\epsilon^6/\nu\right), \end{aligned} \tag{46}$$

and with the zero tangential velocity interface condition by

$$\begin{aligned} u^0_{f,2,k,Q} &\sim C_{k,1}e^{i\widetilde{k}x}\frac{1}{\nu} + O\left(\epsilon^3/\nu\right), \\ u^0_{p,2,k,Q} &\sim P_{k,1}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + P_{k,2}e^{i\widetilde{k}x}\frac{\epsilon^3}{\nu} + P_{k,8}e^{i\widetilde{k}x}\frac{\epsilon^4}{\nu} + O\left(\epsilon^5/\nu\right), \end{aligned} \tag{47}$$

where the coefficients are listed in Appendix A.

Remark. In the quasi-two-dimensional case, one observes the following from (44)–(47).

1. The leading-order terms for the normal velocities for the Stokes–Brinkman and the Stokes–Darcy systems are both of $O(1/\nu)$ in the conduit and are both of $O(\epsilon^2/\nu)$ in the matrix.
2. The normal velocity for the Stokes–Brinkman system contains a boundary layer in the matrix near both $y=0$ and $y=h_p$, the order being $O(\epsilon^2/\nu)$ and $O(\epsilon^4/\nu)$, respectively.
3. The tangential velocities and normal velocities are not independent. Indeed, they are associated with each other via the streamfunctions. Elementary calculations show that the leading order of the tangential velocities in the conduit and matrix are the same as the normal velocities. However, the order of the tangential velocity in the boundary layer changes to $O(\epsilon/\nu)$ and $O(\epsilon^3/\nu)$ at $y=0$ and $y=h_p$, respectively.
4. The normal velocities of Stokes–Darcy equations with Beavers–Joseph interface condition at the interface are different from that with Beavers–Joseph–Saffman–Jones interface condition. The difference is of order $O(\Pi\epsilon^2/\nu)$, which is approximately of order $O(\epsilon^5/\nu)$.

With (44)–(47) in hand, we can compare solutions of the Stokes–Darcy system with different interface conditions with solutions of the Stokes–Brinkman system.

Proposition 4.8. *For the quasi-two-dimensional case, the difference between the normal velocity in the conduit for the Stokes–Brinkman system and the Stokes–Darcy system with the Beavers–Joseph interface condition is of $O(\epsilon^3/\nu)$. That difference is of order $O(\epsilon^2/\nu)$ for the Beavers–Joseph–Saffman–Jones interface condition and is of order $O(\epsilon/\nu)$ for the zero tangential velocity interface condition.*

Proposition 4.9. *In quasi-two-dimensional case, the difference between the normal velocity in the matrix for the Stokes–Brinkman system and the Stokes–Darcy system with the Beavers–Joseph, the Beavers–Joseph–Saffman–Jones, and the zero tangential velocity interface conditions are all of order $O(\epsilon^4/\nu)$.*

Thus, as for the one-dimensional case, we see from Proposition 4.8 that, when comparing with solutions of the Stokes–Brinkman system, the solutions of the Stokes–Darcy system with the Beavers–Joseph interface condition fits better than does the solution obtained using the Beavers–Joseph–Saffman–Jones and both of these fit better than the solution obtained using the zero tangential velocity interface condition.

5. The two-dimensional flows

In Section 4, we obtained, for each wave number $k \geqslant 0$, the quasi-two-dimensional solutions $\vec{u}_{f,k} = (u_{f,1,k}, u_{f,2,k})$ and $\vec{u}_{p,k} = (u_{p,1,k}, u_{p,2,k})$ of the Stokes–Brinkman system in the conduit and matrix, respectively. Likewise, we obtained the corresponding solutions $\vec{u}^0_{f,k,BJ} = (u^0_{f,1,k,BJ}, u^0_{f,2,k,BJ})$ and $\vec{u}^0_{p,k,BJ} = (u^0_{p,1,k,BJ}, u^0_{p,2,k,BJ})$ of the Stokes–Darcy system with the Beavers–Joseph interface condition, $\vec{u}^0_{f,k,BJSJ} = (u^0_{f,1,k,BJSJ}, u^0_{f,2,k,BJSJ})$ and $\vec{u}^0_{p,k,BJSJ} = (u^0_{p,1,k,BJSJ}, u^0_{p,2,k,BJSJ})$ of the Stokes–Darcy system with the Beavers–Joseph–Saffman–Jones interface condition, and $\vec{u}^0_{f,k,Q} = (u^0_{f,1,k,Q}, u^0_{f,2,k,Q})$ and $\vec{u}^0_{p,k,Q} = (u^0_{p,1,k,Q}, u^0_{p,2,k,Q})$ of the Stokes–Darcy system with the zero tangential velocity interface condition. Summation of the quasi-two-dimensional solutions lead to the two-dimensional solutions

$$\vec{u}_f = \sum_{k=-K}^{K} \vec{u}_{f,k}, \qquad \vec{u}_p = \sum_{k=-K}^{K} \vec{u}_{p,k},$$
$$\vec{u}^0_{f,BJ} = \sum_{k=-K}^{K} \vec{u}^0_{f,k,BJ}, \qquad \vec{u}^0_{p,BJ} = \sum_{k=-K}^{K} \vec{u}^0_{p,k,BJ},$$
$$\vec{u}^0_{f,BJSJ} = \sum_{k=-K}^{K} \vec{u}^0_{f,k,BJSJ}, \qquad \vec{u}^0_{p,BJSJ} = \sum_{k=-K}^{K} \vec{u}^0_{p,k,BJSJ},$$
$$\vec{u}^0_{f,Q} = \sum_{k=-K}^{K} \vec{u}^0_{f,k,Q}, \qquad \vec{u}^0_{p,Q} = \sum_{k=-K}^{K} \vec{u}^0_{p,k,Q}.$$

We have the result

$$\left|\vec{u}_{f,k} - \vec{u}^0_{f,k,BJ}\right| \sim \sum_{k=-K}^{K} \left|\vec{u}_{f,k} - \vec{u}^0_{f,k,BJ}\right| \leqslant O\left(\frac{\epsilon^3}{\nu}\right),$$
$$\left|\vec{u}_{f,k} - \vec{u}^0_{f,k,BJSJ}\right| \sim \sum_{k=-K}^{K} \left|\vec{u}_{f,k} - \vec{u}^0_{f,k,BJSJ}\right| \leqslant O\left(\frac{\epsilon^2}{\nu}\right),$$
$$\left|\vec{u}_{f,k} - \vec{u}^0_{f,k,Q}\right| \sim \sum_{k=-K}^{K} \left|\vec{u}_{f,k} - \vec{u}^0_{f,k,Q}\right| \leqslant O\left(\frac{\epsilon}{\nu}\right),$$

which leads to the following conclusion.

Theorem 5.1. *For the two-dimensional case, the difference between the normal velocity in the conduit for the Stokes–Brinkman system and the Stokes–Darcy system with the Beavers–Joseph interface condition is of order $O(\epsilon^3/\nu)$. The difference is of order $O(\epsilon^2/\nu)$ for the Beavers–Joseph–Saffman–Jones interface condition and is of order $O(\epsilon/\nu)$ for the zero tangential velocity interface condition.*

The difference between the normal velocity in the matrix for the Stokes–Brinkman system and Stokes–Darcy system with the Beavers–Joseph, the Beavers–Joseph–Saffman–Jones and the zero tangential velocity interface conditions are all of order $O(\epsilon^4/\nu)$.

Thus, again, the Beavers–Joseph interface condition fits better than the Beavers–Joseph–Saffman–Jones interface condition and both of these fit better than the zero tangential velocity interface condition, when comparing solutions with that of the Stokes–Brinkman system.

6. The convection term in the Brinkman and Darcy equations

The asymptotic analyses of the previous sections can be put to other uses. For example, it can be used to justify neglecting the convection term in the matrix in the Brinkman and Darcy equations.

The steady-state Stokes–Brinkman and Stokes–Darcy equations with convection in the matrix are given by

$$\begin{cases} -\nu\Delta\vec{u}_f + \nabla p_f = \vec{f}_f, \qquad \operatorname{div}\vec{u_f} = 0, & \text{in } \Omega_f, \\ (\vec{u}_p\cdot\nabla)\vec{u}_p - \nu\Delta\vec{u}_p + \dfrac{\nu n}{\Pi}\vec{u}_p + n\nabla p_p - \vec{f}_p = 0, \qquad \operatorname{div}\vec{u_p} = 0, & \text{in } \Omega_p \end{cases} \tag{48}$$

and

$$\begin{cases} -\nu\Delta\vec{u}_f + \nabla p_f = \vec{f}_f, \qquad \operatorname{div}\vec{u_f} = 0, & \text{in } \Omega_f, \\ (\vec{u}_p\cdot\nabla)\vec{u}_p + \dfrac{\nu n}{\Pi}\vec{u}_p + n\nabla p_p - \vec{f}_p = 0, \qquad \operatorname{div}\vec{u_p} = 0, & \text{in } \Omega_p, \end{cases} \tag{49}$$

respectively. We have shown, in the previous sections, that away from the boundary layer, $\vec{u}_p \sim O(\epsilon^2/\nu)$, which is a small quantity in a typical karst aquifer. It is obvious from the expression for $\vec{u}_p$ that the derivatives of the velocities are of $O(\epsilon^2/\nu)$ as well so that the advective term is of $O(\epsilon^4/\nu^2)$. On the other hand, $\frac{\nu n}{\Pi}\vec{u}_p \sim \frac{\nu}{\epsilon^2}\frac{\epsilon^2}{\nu} \sim O(1)$, $\vec{f}_p \sim O(1)$, and $n\nabla p_p \sim O(1)$. In light of these results, we conclude that the advective term is smaller than the others terms and therefore it is justified to neglect it in both the Brinkman and Darcy equation.

7. Conclusion and remarks

We have derived asymptotic solutions with respect to the non-dimensional parameter $\epsilon = \sqrt{\frac{\Pi}{n}}\frac{1}{h_p}$ for the time-independent Stokes–Darcy system with the Beavers–Joseph, Beavers–Joseph–Saffman–Jones, zero tangential velocity, and free-slip interface conditions. The leading order of the velocity is of $O(\frac{1}{\nu})$ in the conduit whereas it is of $O(\frac{\epsilon^2}{\nu})$ in the matrix. It is observed that the optimal choice of the Beavers–Joseph constant α is $\sqrt{n}$ for both the Beavers–Joseph and Beavers–Joseph–Saffman–Jones interface conditions. We also notice that the solutions with the Beavers–Joseph and Beavers–Joseph–Saffman–Jones interface conditions show low sensitivity with respect to α for $\alpha \in [\sqrt{n}, \infty)$. Compared with asymptotic solutions of Stokes–Brinkman system, which we also derived, the solution using the Beavers–Joseph interface condition fits better in the conduit compared to that for the Beavers–Joseph–Saffman–Jones interface condition and both fit better than that obtained using the zero tangential velocity condition; in the matrix, the three choices of conditions yield the same asymptotic behavior. We have also investigated the use of the free-slip interface condition and determined that this is the least accurate among all interface boundary conditions considered here.

In this paper, we only considered the steady-state case and neglected the convection term in the Brinkman and Darcy systems. It would be interesting to examine the time-dependent case. Also, investigating the results when the porosity and permeability are no longer constants would be a challenging and meaningful work.

Appendix A. The normal velocities for the quasi-two-dimensional case

The velocities in the conduit are

$$u^0_{f,2,k,BJ} \sim C_{k,1}e^{i\widetilde{k}x}\frac{1}{\nu} + C_{k,2}e^{i\widetilde{k}x}\frac{\epsilon}{\nu} + C_{k,3}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + O\left(\frac{\epsilon^3}{\nu}\right),$$

$$u^0_{f,2,k,BJSJ} \sim C_{k,1}e^{i\widetilde{k}x}\frac{1}{\nu} + C_{k,2}e^{i\widetilde{k}x}\frac{\epsilon}{\nu} + C_{k,4}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + O\left(\frac{\epsilon^3}{\nu}\right),$$

$$u_{f,2,k} \sim C_{k,1}e^{i\widetilde{k}x}\frac{1}{\nu} + C_{k,2}e^{i\widetilde{k}x}\frac{\epsilon}{\nu} + C_{k,3}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + O\left(\frac{\epsilon^3}{\nu}\right),$$

$$u^0_{f,2,k,Q} \sim C_{k,1}e^{i\widetilde{k}x}\frac{1}{\nu} + O\left(\frac{\epsilon^3}{\nu}\right),$$

where

$$\begin{aligned} C_{k,1} = D_3 &+ \frac{D_2}{2\widetilde{k}(1-4\widetilde{k}h_f e^{-2\widetilde{k}h_f} - e^{-4\widetilde{k}h_f})}\big[h_f\big(1-e^{-2\widetilde{k}h_f}\big)\big(e^{-\widetilde{k}(h_f-y)} - e^{-\widetilde{k}(h_f+y)}\big) \\ &- \big(2\widetilde{k}h_f - 1 + e^{-2\widetilde{k}h_f}\big)ye^{-\widetilde{k}(h_f-y)} - \big(1 - e^{-2\widetilde{k}h_f} - 2\widetilde{k}h_f e^{-2\widetilde{k}h_f}\big)ye^{-\widetilde{k}(h_f+y)}\big] \\ &+ \frac{D_1}{2(1-4\widetilde{k}h_f e^{-2\widetilde{k}h_f} - e^{-4\widetilde{k}h_f})}\big[\big((2+\widetilde{k}h_f) + (2-\widetilde{k}h_f)e^{-2\widetilde{k}h_f}\big)\big(e^{-\widetilde{k}(h_f+y)} - e^{-\widetilde{k}(h_f-y)}\big) \\ &+ \big(3 + 2\widetilde{k}h_f + e^{-2\widetilde{k}h_f}\big)\widetilde{k}ye^{-\widetilde{k}(h_f-y)} + \big(1 + (3-2\widetilde{k}h_f)e^{-2\widetilde{k}h_f}\big)\widetilde{k}ye^{-\widetilde{k}(h_f+y)}\big], \end{aligned}$$

$$C_{k,2} = \frac{D_2 h_p}{1-4\widetilde{k}h_f e^{-2\widetilde{k}h_f} - e^{-4\widetilde{k}h_f}}\big[h_f\big(1+e^{-2\widetilde{k}h_f}\big)(1+y)\big(e^{-\widetilde{k}(h_f-y)} + e^{-\widetilde{k}(h_f+y)}\big)\big]$$
$$+ \frac{D_1 \widetilde{k} h_p}{1-4\widetilde{k}h_f e^{-2\widetilde{k}h_f} - e^{-4\widetilde{k}h_f}}\big[\big((2+\widetilde{k}h_f) - (2-\widetilde{k}h_f)e^{-2\widetilde{k}h_f} + y\widetilde{k}^2\big(1-e^{-2\widetilde{k}h_f}\big)\big)\big(e^{-\widetilde{k}(h_f+y)} - e^{-\widetilde{k}(h_f-y)}\big)\big],$$

$$C_{k,3} = \frac{-if_{p,1,k}(0)h_p^2}{1-4\widetilde{k}h_f e^{-2\widetilde{k}h_f} - e^{-4\widetilde{k}h_f}}\big[-\big(1+e^{-2\widetilde{k}h_f}\big)e^{\widetilde{k}y} + \big(1+4\widetilde{k}h_f + e^{-2\widetilde{k}h_f}\big)e^{-\widetilde{k}(2h_f+y)}$$
$$+ 2\widetilde{k}y\big(e^{\widetilde{k}y} + e^{-\widetilde{k}(2h_f+y)}\big)\big] + h_p^2 \frac{\widetilde{k}f_{p,2,k}(0) + if'_{p,1,k}(0)}{\widetilde{k}(1-4\widetilde{k}h_f e^{-2\widetilde{k}h_f} - e^{-4\widetilde{k}h_f})}\big(\big(1+(1-2\widetilde{k}h_f)e^{-2\widetilde{k}h_f}\big)e^{\widetilde{k}y}$$
$$- \big(1+2\widetilde{k}h_f + e^{-2\widetilde{k}h_f}\big)e^{-\widetilde{k}(2h_f+y)} - \widetilde{k}y\big(1+e^{-2\widetilde{k}h_f}\big)\big(e^{\widetilde{k}y} + e^{-\widetilde{k}(2h_f+y)}\big)\big),$$

$$C_{k,4} = \frac{-if_{p,1,k}(0)h_p^2}{1-4\widetilde{k}h_f e^{-2\widetilde{k}h_f} - e^{-4\widetilde{k}h_f}}\big[-\big(1+(1-2\widetilde{k}h_f)e^{-2\widetilde{k}h_f}\big)e^{\widetilde{k}y} + \big(1+2\widetilde{k}h_f + e^{-2\widetilde{k}h_f}\big)e^{-\widetilde{k}(2h_f+y)}$$
$$+ \widetilde{k}y\big(1+e^{-2\widetilde{k}h_f}\big)\big(e^{\widetilde{k}y} + e^{-\widetilde{k}(2h_f+y)}\big)\big] + h_p^2 \frac{\widetilde{k}f_{p,2,k}(0) + if'_{p,1,k}(0)}{\widetilde{k}(1-4\widetilde{k}h_f e^{-2\widetilde{k}h_f} - e^{-4\widetilde{k}h_f})}$$
$$\times \big(\big(1+(1-2\widetilde{k}h_f)e^{-2\widetilde{k}h_f}\big)e^{\widetilde{k}y} - \big(1+2\widetilde{k}h_f + e^{-2\widetilde{k}h_f}\big)e^{-\widetilde{k}(2h_f+y)} - \widetilde{k}y\big(1+e^{-2\widetilde{k}h_f}\big)\big(e^{\widetilde{k}y} + e^{-\widetilde{k}(2h_f+y)}\big)\big),$$

$$D_1 = \frac{1}{4\widetilde{k}^2}\Bigg[e^{-\widetilde{k}h_f}\int_0^{-h_f} e^{-\widetilde{k}s}(1+\widetilde{k}s)\big(\widetilde{k}f_{f,2,k}(s) + if'_{f,1,k}(s)\big)\,ds + e^{\widetilde{k}h_f}\int_0^{-h_f} e^{\widetilde{k}s}(-1+\widetilde{k}s)\big(\widetilde{k}f_{f,2,k}(s) + if'_{f,1,k}(s)\big)\,ds$$
$$+ h_f e^{-\widetilde{k}h_f}\int_0^{-h_f} e^{-\widetilde{k}s}\widetilde{k}\big(\widetilde{k}f_{f,2,k}(s) + if'_{f,1,k}(s)\big)\,ds + h_f e^{\widetilde{k}h_f}\int_0^{-h_f} e^{\widetilde{k}s}\widetilde{k}\big(\widetilde{k}f_{f,2,k}(s) + if'_{f,1,k}(s)\big)\,ds\Bigg],$$

$$D_2 = \frac{1}{4}\Bigg[-e^{-\widetilde{k}h_f}\int_0^{-h_f} e^{-\widetilde{k}s}(1-\widetilde{k}s)\big(\widetilde{k}f_{f,2,k}(s) + if'_{f,1,k}(s)\big)\,ds + e^{\widetilde{k}h_f}\int_0^{-h_f} e^{\widetilde{k}s}(1+\widetilde{k}s)\big(\widetilde{k}f_{f,2,k}(s) + if'_{f,1,k}(s)\big)\,ds$$
$$+ h_f e^{-\widetilde{k}h_f}\int_0^{-h_f} e^{-\widetilde{k}s}\widetilde{k}\big(\widetilde{k}f_{f,2,k}(s) + if'_{f,1,k}(s)\big)\,ds + h_f e^{\widetilde{k}h_f}\int_0^{-h_f} e^{\widetilde{k}s}\widetilde{k}\big(\widetilde{k}f_{f,2,k}(s) + if'_{f,1,k}(s)\big)\,ds\Bigg],$$

$$D_3 = \frac{1}{4\widetilde{k}^2}\Bigg[e^{\widetilde{k}y}\int_0^{y} e^{-\widetilde{k}s}(1+\widetilde{k}s)\big(\widetilde{k}f_{f,2,k}(s) + if'_{f,1,k}(s)\big)\,ds + e^{-\widetilde{k}y}\int_0^{y} e^{\widetilde{k}s}(-1+\widetilde{k}s)\big(\widetilde{k}f_{f,2,k}(s) + if'_{f,1,k}(s)\big)\,ds$$
$$+ h_f e^{\widetilde{k}y}\int_0^{y} e^{-\widetilde{k}s}\widetilde{k}\big(\widetilde{k}f_{f,2,k}(s) + if'_{f,1,k}(s)\big)\,ds + h_f e^{-\widetilde{k}y}\int_0^{y} e^{\widetilde{k}s}\widetilde{k}\big(\widetilde{k}f_{f,2,k}(s) + if'_{f,1,k}(s)\big)\,ds\Bigg].$$

The velocities in the porous media are

$$u^0_{p,2,k,BJ} \sim P_{k,1}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + P_{k,2}e^{i\widetilde{k}x}\frac{\epsilon^3}{\nu} + P_{k,3}e^{i\widetilde{k}x}\frac{\epsilon^4}{\nu} + O\left(\frac{\epsilon^6}{\nu}\right),$$
$$u^0_{p,2,k,BJSJ} \sim P_{k,1}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + P_{k,2}e^{i\widetilde{k}x}\frac{\epsilon^3}{\nu} + P_{k,3}e^{i\widetilde{k}x}\frac{\epsilon^4}{\nu} + P_{k,7}e^{i\widetilde{k}x}\frac{\Pi\epsilon^2}{\nu} + O\left(\frac{\epsilon^6}{\nu}\right),$$
$$u_{p,2,k} \sim P_{k,1}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + P_{k,2}e^{i\widetilde{k}x}\frac{\epsilon^3}{\nu} + P_{k,4}e^{i\widetilde{k}x}\frac{\epsilon^4}{\nu} + O\left(\frac{\epsilon^5}{\nu}\right) + P_{k,5}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu}e^{-Ey} + P_{k,6}e^{i\widetilde{k}x}\frac{\epsilon^4}{\nu}e^{E(y-h_p)},$$
$$u^0_{p,2,k,Q} \sim P_{k,1}e^{i\widetilde{k}x}\frac{\epsilon^2}{\nu} + P_{k,2}e^{i\widetilde{k}x}\frac{\epsilon^3}{\nu} + P_{k,8}e^{i\widetilde{k}x}\frac{\epsilon^4}{\nu} + O\left(\frac{\epsilon^6}{\nu}\right),$$

where

$$P_{k,1} = -\frac{h_p^2}{\widetilde{k}}\big(\widetilde{k} f_{p,2,k}(h_p) + i f'_{p,1,k}(h_p)\big)e^{\widetilde{k}(y-h_p)} + \frac{h_p^2}{\widetilde{k}}\big(\widetilde{k} f_{p,2,k}(y) + i f'_{p,1,k}(y)\big) + i f_{p,1,k}(0)e^{-\widetilde{k}y}h_p^2,$$
$$P_{k,2} = 2\widetilde{k} i f_{p,1,k}(0)e^{-\widetilde{k}y}h_p^3 - i f_{f,1,k}(0)e^{-\widetilde{k}y}h_p^2 + 4\widetilde{k}^3 h_p^3\big((2+\widetilde{k}h_f) - (2-\widetilde{k}h_f)e^{-2\widetilde{k}h_f}\big)D_1 e^{-\widetilde{k}(y+h_f)}$$
$$- 4\widetilde{k}^2 h_p^3 h_f\big(1+e^{-2\widetilde{k}h_f}\big)D_2 e^{-\widetilde{k}(y+h_f)},$$
$$P_{k,3} = -2\widetilde{k}^2 i f_{f,1,k}(0)e^{-\widetilde{k}y}h_p^3 + 4\widetilde{k}^4 h_p^4\big((2+\widetilde{k}h_f) - (2-\widetilde{k}h_f)e^{-2\widetilde{k}h_f}\big)D_1 e^{-\widetilde{k}(y+h_f)}$$
$$- 4\widetilde{k}^3 h_p^4 h_f\big(1+e^{-2\widetilde{k}h_f}\big)D_2 e^{-\widetilde{k}(y+h_f)},$$
$$P_{k,4} = \frac{8h_f\widetilde{k}^3 i f_{p,1,k}(0)e^{-2\widetilde{k}^2 h_f}h_p^4}{1-4\widetilde{k}h_f e^{-2\widetilde{k}h_f} - e^{-4\widetilde{k}h_f}} - 2\widetilde{k} i f_{f,1,k}(0)e^{-\widetilde{k}y}h_p^3 + 4\widetilde{k}^4 h_p^4\big(1-e^{-2\widetilde{k}h_f}\big)D_1 e^{-\widetilde{k}(y+h_f)}$$
$$- 4\widetilde{k}^2 h_p^4\big(1+e^{-2\widetilde{k}h_f}\big)D_2 e^{-\widetilde{k}(y+h_f)},$$
$$P_{k,5} = \frac{2\widetilde{k}^2 h_p^2}{1-4\widetilde{k}h_f e^{-2\widetilde{k}h_f} - e^{-4\widetilde{k}h_f}}\big((1+\widetilde{k}h_f) - (1-\widetilde{k}h_f)e^{-2\widetilde{k}h_f}\big)D_1 e^{-\widetilde{k}h_f}$$
$$+ \frac{2h_p^2}{1-4\widetilde{k}h_f e^{-2\widetilde{k}h_f} - e^{-4\widetilde{k}h_f}}\big((1-\widetilde{k}h_f) - (1+\widetilde{k}h_f)e^{-2\widetilde{k}h_f}\big)D_2 e^{-\widetilde{k}h_f},$$
$$P_{k,6} = -\widetilde{k}h_p^4\big(\widetilde{k} f_{p,2,k}(y) + i f'_{p,1,k}(y)\big),$$
$$P_{k,7} = -\frac{8\widetilde{k}^4 h_p^5 h_f i f_{f,1,k}(0)e^{-2\widetilde{k}h_f}}{1-4\widetilde{k}h_f e^{-2\widetilde{k}h_f} - e^{-4\widetilde{k}h_f}},$$
$$P_{k,8} = -2\widetilde{k}^2 i f_{f,1,k}(0)e^{-\widetilde{k}y}h_p^3,$$

where D_1 and D_2 are given as above.

References

[1] J. Bear, Dynamics of Fluids in Porous Media, Dover, 1988.
[2] G. Beavers, D. Joseph, Boundary conditions at a naturally permeable wall, J. Fluid Mech. 30 (1967) 197–207.
[3] Y. Cao, M. Gunzburger, F. Hua, X. Wang, Coupled Stokes–Darcy model with Beavers–Joseph interface boundary condition, Commun. Math. Sci. 8 (1) (March 2010) 1–25.
[4] Y. Cao, M. Gunzburger, X. Hu, F. Hua, X. Wang, W. Zhao, Finite element approximations for Stokes–Darcy flow with Beavers–Joseph interface conditions, SIAM J. Numer. Anal. 47 (2010) 4239–4256.
[5] H. Brinkman, A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles, Appl. Sci. Res. A 1 (1947) 27–34.
[6] M. Discacciati, E. Miglio, A. Quarteroni, Mathematical and numerical models for coupling surface and groundwater flows, Appl. Numer. Math. 1 (2002) 57–74.
[7] M. Discacciati, A. Quarteroni, Analysis of a domain decomposition method for the coupling of the Stokes and Darcy equations, in: F. Brezzi, et al. (Eds.), Numerical Mathematics and Advanced Applications. Proceedings of ENUMATH 2001, Springer, Milan, 2003, pp. 3–20.
[8] M. Discacciati, A. Quarteroni, Convergence analysis of a subdomain iterative method for the finite element approximation of the coupling of Stokes and Darcy equations, Comput. Vis. Sci. 6 (2004) 93–103.
[9] F. Hua, Modeling, analysis and simulation of Stokes–Darcy system with Beavers–Joseph interface condition, PhD thesis, Florida State University, 2009.
[10] W. Jäger, A. Mikelić, On the boundary condition at the contact interface between a porous medium and a free fluid, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 23 (1996) 403–1465.
[11] W. Jäger, A. Mikelić, On the interface boundary condition of Beavers, Joseph, and Saffman, SIAM J. Appl. Math. 60 (2000) 1111–1127.
[12] B. Jiang, A parallel domain decomposition method for coupling of surface and groundwater flows, Comput. Methods Appl. Mech. Engrg. 198 (2009) 947–957.
[13] I. Jones, Reynolds number flow past a porous spherical shell, Proc. Cambridge Philos. Soc. 73 (1973) 231–238.
[14] W. Layton, F. Schieweck, I. Yotov, Coupling fluid flow with porous media flow, SIAM J. Numer. Anal. 40 (2003) 2195–2218.
[15] G. Le Bars, M. Worster, Interfacial conditions between a pure fluid and a porous medium implications for binary alloy solidification, J. Fluid Mech. 550 (2006) 149–173.
[16] J.-L. Lions, Perturbations singulières dans les problèmes aux limites et en contrôle optimal, Springer-Verlag, Berlin, New York, 1973.
[17] W. McCabe, J. Smith, P. Harriot, Unit Operations of Chemical Engineering, seventh ed., McGraw–Hill, New York, 2005.
[18] G. Neale, W. Nader, Practical significance of Brinkman's extension of Darcy's law: coupled parallel flows within a channel and a bounding porous, Canad. J. Chem. Eng. 52 (1974) 475–478.
[19] P. Saffman, On the boundary condition at the interface of a porous medium, Stud. Appl. Math. 1 (1971) 77–84.